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UNIVERSITY OF ILLINOIS

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COLUMBIA UNIVERSITY

D. C. LEWIS, JR.
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# GENERALIZED GALOIS THEORY FOR RINGS WITH MINIMUM CONDITION.*

By TADASI NAKAYAMA.

In a previous note [1] the writer established a Galois theory for general rings with minimum condition which specializes in the case of division rings to Jacobson's Galois theory with outer groups of automorphisms.[2] On the other hand, Jacobson and Cartan have, independently, given a generalized Galois theory for division rings which embraces the theory of that type and the theory of commuter systems.[3] The present work is to propose a similar generalization for rings with minimum condition, or primary rings, to be exact.[4] This we shall do by combining the results in the above paper with those in another note,[5] in which the writer studied commuter systems in general rings with minimum condition.

**0.** Let $R$ be,[6] throughout in this note, a ring with unit element 1 and satisfying the minimum condition (whence the maximum condition) for ideals. Let $\mathfrak{A}$ be the (absolute) endomorphism ring of the module $R$. The left and right multiplication rings $R_l$, $R_r$ are subrings of $\mathfrak{A}$. Let $\mathfrak{B}$ be a subring of $\mathfrak{A}$ that contains $R_r$; $\mathfrak{B} \supseteq R_r$. Its commuter system $V_{\mathfrak{A}}(\mathfrak{B})$ in $\mathfrak{A}$ is a subring of $V_{\mathfrak{A}}(R_r) = R_l$, and has a form $V_{\mathfrak{A}}(\mathfrak{B}) = S_l$, the ring of left multiplications on $R$ of $S$, with a subring $S$ of $R$.

Suppose that $R$ is (right-) *regular* with respect to $\mathfrak{B}$, in the sense of [8] or [9].[7] Then $R$ is regular with respect to $S_l = V_{\mathfrak{A}}(\mathfrak{B})$ too, and

* Received March 8, 1949; revised June 14, 1950. The writer is very grateful to the referee, whose kind comments were very useful, as well as valuable, for the revision.

[1] [8].

[2] [3]. Cf. also [1], [7].

[3] [2], [5].

[4] Theorem 2 in § 7 below.

[5] [9].

[6] Together with every ring which we shall consider, except perhaps $\mathfrak{A}$ below. Further, we consider only those subrings of $R$ which contain the unit element of $R$.

[7] [8], § 1; or [9], § 0. A right-module $\mathfrak{m}$ of $R$ is called *regular*, when a direct sum of a certain number, say $m$, of its copies is isomorphic to the direct sum of a certain number, say $n$ ($\neq 0$), of copies of $R$ itself, that is, $\mathfrak{m}^m \simeq R^n$. The number $k = n/m$, uniquely determined by $\mathfrak{m}$, is called its rank. We have: $\alpha$) a right-regular module $\mathfrak{m}$ of $R$ is also regular with respect to the ($R$-)endomorphism ring $R^*$ with rank $k^{-1}$; $\beta$) the $R^*$-endomorphism ring of $\mathfrak{m}$ coincides with $R$; $\gamma$) if $\mathfrak{m}$, of rank $k$,

$V_{\mathfrak{A}}(S_l) = \mathfrak{B}$. If, conversely, $S$ is a subring of $R$ and $R$ is right-regular with respect to $S_l$ (that is, $R$ is left-regular with respect to $S$), then $\mathfrak{B} = V_{\mathfrak{A}}(S_l)$ is a subring of $\mathfrak{A}$, containing $R_r$, with respect to which $R$ is regular. *So the* $\mathfrak{B}$ *and* $S$ *of these types correspond to each other* 1-1 *dually.* The $R_r$-rank of $\mathfrak{B}$ is equal to the $S_l$-rank of $R$.

Another notion which the writer wants to take from his former paper is that of a *Galois class-group.*[8] Consider a finite group of automorphism-classes of $R$, class referring to the inner automorphism group. Let $\rho_1, \rho_2, \cdots, \rho_g$ be its representative system. To each $\rho = \rho_i$ we may define an $R$-double-module $(\rho, R)$ which is $R$ itself as $R$-right-module and for which the left-operation of an element $a$ in $R$ is explained by the left-multiplication of $a^\rho$, that is, by $(a^\rho)_l$; it is determined uniquely up to isomorphism by the class of $\rho$. We say that our finite group of automorphism-classes, represented by $\rho_1, \rho_2, \cdots, \rho_g$, forms a Galois class-group if no $(\rho_i, R)$ and $(\rho_j, R)$ with $i \neq j$ possess mutually $R$-two-sided isomorphic composition residue-moduli.[9] In this case, each submodule of the direct sum $(\rho_1, R) + (\rho_2, R) + \cdots + (\rho_g, R)$ is a direct sum of submoduli of $(\rho_i, R)$, $i = 1, 2, \cdots, g$, according to the well known theorem of the decomposition of a group into *idealistic* components.[10]

**1.** Now let $\mathfrak{B}_0$ be a subring of $\mathfrak{A}$ containing $R_r$ and satisfying the minimum condition. Let further $G = \{1, \sigma, \cdots, \tau\}$ be a finite group of order $g$, and $\rho(1), \rho(\sigma), \cdots, \rho(\tau)$ automorphisms of $R$ such that $\rho(\sigma)\rho(\tau)\rho(\sigma\tau)^{-1} \in \mathfrak{B}_0$ for every pair $\sigma, \tau$.[11] Each $\rho(\tau)$, as well as any automorphism $\rho$ of $R$, induces an automorphism in $\mathfrak{A}$ in natural manner: $x^\rho \alpha^\rho = (x\alpha)^\rho$ $(x \in R)$, or $\alpha^\rho = \rho^{-1}\alpha\rho$. We have $(a_l)^\rho = (a^\rho)_l$, $(a_r)^\rho = (a^\rho)_r$ (which depend essentially on the ring-automorphism property of $\rho$). Assume that in this sense $\rho(1), \rho(\sigma), \cdots, \rho(\tau)$ induce automorphisms in $\mathfrak{B}_0$, that is, $\rho(\sigma)^{-1}\mathfrak{B}_0\rho(\sigma) = \mathfrak{B}_0$ for every $\sigma \in G$, and moreover that $\{\rho(1), \rho(\sigma), \cdots, \rho(\tau)\}$ forms, again in the

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is also regular and of rank $l$ with respect to a subring $T$ of $R$, then any other regular module of $R$ is regular with respect to $T$ too and its $T$-rank is equal to $lk^{-1}$ times its $R$-rank.

[8] [8], § 3.

[9] If $R$ is a division ring, or more generally a simple ring, this condition is implied automatically from the fact that we are dealing with a group of automorphism-*classes.*

[10] A direct decomposition of a group is called *idealistic,* if no two distinct components have isomorphic composition factor-groups and if the decomposition is the finest as such. We propose to use "idealistic" (rather than "ideal") not to be mistaken for "ideal" in a ring.

[11] $\rho(\sigma)\rho(\tau)\rho(\sigma\tau)^{-1}$ are regular elements in $\mathfrak{B}_0$, since they are non-zero-divisors, and $\mathfrak{B}_0$ is a ring with unit element and minimum condition.

same sense, a representative system of a Galois class-group of the ring $\mathfrak{B}_0$ (rather than $R$).

Let $\mathfrak{B}$ be the ring of endomorphisms of $R$ generated by $\rho(1), \rho(\sigma), \cdots, \rho(\tau)$ and $\mathfrak{B}_0$, that is,

$$\mathfrak{B} = \rho(1)\mathfrak{B}_0 + \rho(\sigma)\mathfrak{B}_0 + \cdots + \rho(\tau)\mathfrak{B}_0.$$

Here $\rho(1)\mathfrak{B}_0, \rho(\sigma)\mathfrak{B}_0, \cdots, \rho(\tau)\mathfrak{B}_0$ are $\mathfrak{B}_0$-two-sided isomorphic to the moduli $(\rho(1), \mathfrak{B}_0), (\rho(\sigma), \mathfrak{B}_0), \cdots, (\rho(\tau), \mathfrak{B}_0)$ in our above notation, $\rho(1), \rho(\sigma), \cdots, \rho(\tau)$ being viewed as automorphisms of $\mathfrak{B}_0$. Hence our assumed Galois class-group property implies that the sum is direct, and $\mathfrak{B}$ is a so-called crossed product of $\{\rho(1), \rho(\sigma), \cdots, \rho(\tau)\}$ and $\mathfrak{B}_0$. Further, [8], Lemma 5 [11] can be applied to $\mathfrak{B}$. Namely, if two right-moduli of $\mathfrak{B}$ are $\mathfrak{B}_0$-regular and mutually $\mathfrak{B}_0$-isomorphic, then they are $\mathfrak{B}$-isomorphic.

Now suppose that $R$ is regular, of rank $b = n/m$, with respect to $\mathfrak{B}_0$. Then the $\mathfrak{B}$-module $R^{mg}$ is ($\mathfrak{B}_0$-, whence) $\mathfrak{B}$-isomorphic to $\mathfrak{B}^n$, by virtue of our fundamental lemma just cited, and $R$ is $\mathfrak{B}$-regular with rank $bg^{-1}$. Thus § 0 can be applied to $\mathfrak{B}$; $V_{\mathfrak{A}}(\mathfrak{B}) = S_l$ $(S \subseteq R)$, $V_{\mathfrak{A}}(S_l) = \mathfrak{B}$ and $R$ is $S_l$-regular with rank $b^{-1}g$. Hence we have

THEOREM 1. *Let $\mathfrak{B}_0$ be a subring (satisfying minimum condition) of the absolute endomorphism ring $\mathfrak{A}$ of $R$ containing $R_r$ such that $R$ is $\mathfrak{B}_0$-regular of rank $b$. Let $G = \{1, \sigma, \cdots, \tau\}$ be a finite group and let a system $\{\rho(1), \rho(\tau), \cdots, \rho(\tau)\}$ of (ring-)automorphisms of $R$ be such that $\rho(\sigma)^{-1}\mathfrak{B}_0\rho(\sigma) = \mathfrak{B}_0$, $\rho(\sigma)\rho(\tau)\rho(\sigma\tau)^{-1} \varepsilon \mathfrak{B}_0$ for every $\sigma, \tau \varepsilon G$ and form a representative system of a Galois class-group of $\mathfrak{B}_0$. Then $R$ is regular of rank $bg^{-1}$ with respect to $\mathfrak{B} = \Sigma\rho(\sigma)\mathfrak{B}_0$, and the commuter system $V_{\mathfrak{A}}(\mathfrak{B})$ (i. e., the $\mathfrak{B}$-endomorphism ring of $R$) is $S_l$ with a subring $S$ of $R$ such that $R$ is $S$-left-regular over $S$ with rank $b^{-1}g$, $g$ being the order of $G$. Further $V_{\mathfrak{A}}(S_l) = \mathfrak{B}$.*

**2.** (Our above Galois class-group condition, on the system $\{\rho(1), \rho(\sigma), \cdots, \rho(\tau)\}$ refers to $\mathfrak{B}_0$ rather than to $R$. Now we want to replace it by an assumption which refers to the latter, i. e. to $R$ itself.) Let $\mathfrak{B}_0$ be a

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[12] Namely: let $X$ be a ring with unit element and minimum condition and let $Y$ be its radical. Let a group $\mathfrak{G}$ of automorphism-classes of $X$ induce a Galois class-group in the residue-ring $X/Y$ (which is weaker that it forms one of $X$ itself). Consider a crossed product $(\mathfrak{G}, X)$ (with factor set) and two $(\mathfrak{G}, X)$-right-moduli $\mathfrak{s}$, $\mathfrak{t}$, which are direct sums of $X$-submoduli $X$-isomorphic to directly indecomposable right-ideal components of $X$. If then $\mathfrak{s}/\mathfrak{s}Y$ and $\mathfrak{t}/\mathfrak{t}Y$ *are $X$-isomorphic, $\mathfrak{s}$ and $\mathfrak{t}$ are $(\mathfrak{G}, X)$-isomorphic.* Note that the residue-moduli $\mathfrak{s}/\mathfrak{s}Y$, $\mathfrak{t}/\mathfrak{t}Y$ in the assumption are replaced by the moduli $\mathfrak{s}$, $\mathfrak{t}$ themselves in the conclusion, and which is more important, the operator domain $X$ by $(\mathfrak{G}, X)$. For a special case cf. [7], Lemma 1.

subring of $\mathfrak{A}$ which contains $R_r$ and satisfies the double chain as $R_r$-two-sided module. Assume that every composition residue-module of $\mathfrak{B}_0$, as $R_r$-two-sided module, is isomorphic to one of $R_r$. Let a finite system $\{\rho(1), \rho(\sigma), \cdots, \rho(\tau)\}$ of automorphisms of $R$, given in connection with a group $G = \{1, \sigma, \cdots, \tau\}$, be such that $\rho(\sigma)^{-1}\mathfrak{B}_0\rho(\sigma) = \mathfrak{B}_0$, $\rho(\sigma)\rho(\tau)\rho(\sigma\tau)^{-1} \varepsilon \mathfrak{B}_0$, again, and moreover that the $R_r$-two-sided moduli $\rho(\sigma)R_r$, $\rho(\tau)R_r$ with distinct $\sigma, \tau$ have no isomorphic composition residue-moduli. Then the system $\{\rho(\tau)\}$ certainly forms a representative system of a Galois class-group of $\mathfrak{B}_0$, $\rho(\sigma)$ being viewed as automorphisms of $\mathfrak{B}_0$.

**3.** If in particular $\mathfrak{B}_0 \subseteq R_rR_l$ and $\mathfrak{B}_0$ is generated by a finite number of elements in $R_l$ over $R_r$, then the condition we imposed on $\mathfrak{B}_0$ in the preceding section is fulfilled. We have $\mathfrak{B}_0 = R_rK_l$ with a subring $K$ of $R$; indeed, with $K_l = \mathfrak{B}_0 \cap R_l$.[13] Moreover $K_l = (\Sigma\rho(\sigma)\mathfrak{B}_0) \cap R_l$, too,[14] where of course $\{\rho(\sigma)\}$ is as in § 2. The subring $S$ of $R$ given by $S_l = V_{\mathfrak{A}}(\mathfrak{B}) = V_{\mathfrak{A}}(\Sigma\rho(\sigma)\mathfrak{B}_0)$ is characterized as the totality of elements of $R$ which are invariant under $\{(\rho), \rho(\sigma), \cdots, \rho(\tau)\}$ and which are commutative with $K$ elementwise. For,

$$\begin{aligned} V_{\mathfrak{A}}(\mathfrak{B}) &= V_{\mathfrak{A}}(R_r) \cap V_{\mathfrak{A}}(K_l) \cap V_{\mathfrak{A}}(\{\rho(1), \rho(\sigma), \cdots, \rho(\tau)\}) \\ &= R_l \cap V_{\mathfrak{A}}(K_l) \cap V_{\mathfrak{A}}(\{\rho(1), \rho(\sigma), \cdots, \rho(\tau)\}) \\ &= V_{R_l}(K_l) \cap V_{\mathfrak{A}}(\{\rho(1), \rho(\sigma), \cdots, \rho(\tau)\}). \end{aligned}$$

If, further $R$ is regular with respect to $\mathfrak{B}_0 = R_rK_l$, then ($\mathfrak{B} = V_{\mathfrak{A}}(S_l)$ and so $K_l = \mathfrak{B}_0 \cap R_l = \mathfrak{B} \cap R_l = V_{\mathfrak{A}}(S_l) \cap R_l = V_{R_l}(S_l)$ whence) $K = V_R(S)$. We state these facts as

LEMMA 1. *Let a subring $\mathfrak{B}_0$ of $\mathfrak{A}$ contain $R_r$, satisfy the $R_r$-two-sided double chain condition, and moreover be such that every composition residue-module of $\mathfrak{B}_0$, as $R_r$-two-sided module, is isomorphic to one of $R_r$. Let a system $\{\rho(\tau)\}$ of automorphisms of $R$, with a finite group $G = \{\sigma\}$, be such that $\rho(\sigma)^{-1}\mathfrak{B}_0(\sigma) = \mathfrak{B}_0$, $\rho(\sigma)\rho(\tau)\rho(\sigma\tau)^{-1} \varepsilon \mathfrak{B}_0$, for all $\sigma, \tau \varepsilon G$, and that $(\rho(\sigma), R)$ and $(\rho(\tau), R)$ with distinct $\sigma, \tau$, in the notation of § 0, have no isomorphic composition residue-moduli. Assume further that $R$ is $\mathfrak{B}_0$-regular. Then Theorem 1 may be applied to $\mathfrak{B}_0$ and $\{\rho(\sigma)\}$. Our assumption on the $R_r$-behavior of $\mathfrak{B}_0$ is satisfied in particular if $\mathfrak{B}_0$ is contained in $R_rR_l$ and is generated by a finite number of elements in $R_l$ over $R_r$. If here we put $K_l = \mathfrak{B}_0 \cap R_l$, then $\mathfrak{B}_0 = R_rK_l$, and $\rho(\sigma)^{-1}\mathfrak{B}_0\rho(\sigma) = \mathfrak{B}_0$ is equivalent to*

[13] $K$ contains the center of $R$.

[14] Cf. again the theorem of idealistic components.

$K^{\rho(\sigma)} = K$. *Moreover* $K = V_R(S)$, *where* $S$ *is as in Theorem* 1 *and is in fact the totality of element of* $V_R(K)$ *invariant under* $\rho(1), \rho(\sigma), \cdots, \rho(\tau)$.

If we suppose that a finite set $a_1, a_2, \cdots, a_k$ of elements of $K$ can be so chosen that the left-multiplications $a_{1l}, a_{2l}, \cdots, a_{kl}$, on $R$, form an independent basis of $\mathfrak{B}_0 = R_r K_l$ over $R_r$, then $a_1, a_2, \cdots, a_k$ form a basis of $K$ over the center $Z$ of $R$. For, an element $x_{1r}a_{1l} + x_{2r}a_{2l} + \cdots + x_{kr}a_{kl}$ $(x_i \in R)$ belongs to $R_l$ if and only if every $x_i$ is in $Z$. The converse is also valid as we readily see by a similar argument. And, if such is the case, $\mathfrak{B}_0$ is certainly right- (as well as left-) regular over $R_r$, with rank $k$. Hence, if $R$ is $\mathfrak{B}_0$-right-regular, its rank $b$ is $k^{-1}$, and $R^{kg}$ is $\mathfrak{B}$-isomorphic to $\mathfrak{B}$, provided of course that our additional conditions are fulfilled. $R$ possesses an independent left-basis over $S$ with $kg$ terms.

**4.** Let $\mathfrak{B}_0$, $\rho(\sigma)$ be as in § **1**, Theorem 1. Consider an endomorphism $\phi$ of $R$, as module, which commutes elementwise with $S_l$: $\phi \in V_{\mathfrak{R}}(S_l) = \mathfrak{B}$ $= \Sigma\rho(\sigma)\mathfrak{B}_0$. If $\phi\mathfrak{B}_0 = \mathfrak{B}_0\phi$ and this $\mathfrak{B}_0$-two-sided module is directly indecomposable, then it is contained in an idealistic component of (the $\mathfrak{B}_0$-two-sided module) $\mathfrak{B}$, whence in one of $\rho(\sigma)\mathfrak{B}_0$. Here the second assumption is satisfied, either when $\phi$ is an automorphism of $R$ as module and $\mathfrak{B}_0$ is (two-sided) directly indecomposable, or when $\mathfrak{B}_0$ is primary, provided that the first assumption $\phi\mathfrak{B}_0 = \mathfrak{B}_0\phi$ is fulfilled.

Suppose next that $\mathfrak{B}_0 = R_r K_l$ $(K_l = R_l \cap \mathfrak{B}_0)$ as in § **3** and that $\phi$ is an automorphism of the ring $R$ (as a ring) which commutes with $S_l$ elementwise, or, which is the same, leaves $S$ elementwise invariant. Then $K^{\phi} = K$, since $K = V_R(S)$, and $\phi^{-1}K_l\phi = K_l$. Thus $\phi^{-1}\mathfrak{B}_0\phi = \mathfrak{B}_0$ and the former of our conditions, $\phi\mathfrak{B}_0 = \mathfrak{B}_0\phi$, is satisfied automatically. Thus

LEMMA 2. *Let* $\mathfrak{B}_0$, $\rho(\sigma)$, $S$ *be as in Theorem* 1. *Let* $\phi \in \mathfrak{B} = V_{\mathfrak{R}}(S_l)$. *If* $\phi\mathfrak{B}_0 = \mathfrak{B}_0\phi$ *and* $\phi\mathfrak{B}_0$ *is* $\mathfrak{B}_0$*-two-sided directly indecomposable, then* $\phi$ *is contained in one of the* $\rho(\sigma)\mathfrak{B}_0$. *Here the second part of the last assumption, i. e. the indecomposability of* $\phi\mathfrak{B}_0$, *is satisfied either when* $\mathfrak{B}_0$ *is two-sided directly indecomposable and* $\phi$ *is an automorphism of* $R$ *as module, or when* $\mathfrak{B}_0$ *is primary, provided that the first part is fulfilled. Further, if* $\mathfrak{B}_0 = R_r K_l$, *as in the second half of Lemma* 1, *then automatically* $\phi\mathfrak{B}_0 = \mathfrak{B}_0\phi$ *for every automorphism* $\phi$ *of* $R$ *as ring leaving* $S$ *elementwise invariant.*

Now, let $\Phi$ and $\Phi_0$ be the totalities of automorphisms of $R$ (as a ring) which are contained in $\mathfrak{B}$ and $\mathfrak{B}_0$, respectively. They are groups, since $\mathfrak{B}$ and $\mathfrak{B}_0$ are rings with unit element and satisfying minimum condition. If $\mathfrak{B}_0 = R_r K_l$ $(K_l = R_l \cap \mathfrak{B}_0)$ then $\Phi_0$ is invariant in $\Phi$, because $\phi K_l = K_l\phi$,

$\phi\mathfrak{B}_0 = \mathfrak{B}_0\phi$ for every $\phi \in \Phi$, as was just observed. If further $\mathfrak{B}_0 = R_rK_l$ is two-sided directly indecomposable, then every $\phi \in \Phi$ is contained in a certain $\rho(\sigma)\mathfrak{B}_0$, whence in $\rho(\sigma)\Phi_0$. So

LEMMA 3. *If again $\mathfrak{B}_0 = R_rK_l$ (and $R$ is $\mathfrak{B}_0$-regular) then $\Phi_0$ is an invariant subgroup of the group $\Phi$. If further $\mathfrak{B}_0 = R_rK_l$ is two-sided directly indecomposable, $\{\rho(\sigma)\}$ forms a representative system of cosets in $\Phi$ with respect to the invariant subgroup $\Phi_0$, and the factor-group $\Phi/\Phi_0$ is isomorphic to $G$.*

Let it be also observed that if $K$ has the property that every element of $K$ may be expressed as a sum of regular elements, then $R_rK_l = R_r\Phi_0$ [15] and indeed $R_rK_l = R_r\Phi_1$ where $\Phi_1$ is the totality of inner automorphisms $a_ra_l^{-1}$ of $R$ generated by regular elements $a$ in $K$. This last property is enjoyed by a very wide class of rings [16]; we note here for later use, that primary rings certainly have the proposed property.

**5.** Now we wish to study a correspondence between certain subrings of $\mathfrak{B}$ and those of $R$. Let $\mathfrak{B}_0 = R_rK_l$, $\{\rho(\sigma)\}$ and $S$ be as in §§ **2, 3.** Let $H = \{1, \alpha, \cdots, \beta\}$ be a subgroup of $G$, and let there be given for each $\alpha \in H$ an automorphism $\rho_1(\alpha)$ of $R$ contained in $\rho(\alpha)\mathfrak{B}_0$ (whence in $\rho(\alpha)\Phi_0$). Let further $\mathfrak{B}_1$ be a subring of $\mathfrak{B}_0$ such that $\mathfrak{B}_1 \supseteq R_r$, $\rho_1(\alpha)^{-1}\mathfrak{B}_1\rho_1(\alpha) = \mathfrak{B}_1$, $\rho_1(\alpha)\rho_1(\beta)\rho_1(\alpha\beta)^{-1} \in \mathfrak{B}_1$ for all $\alpha, \beta \in H$. Obviously the ring $\mathfrak{B}^{\S} = \Sigma\rho_1(\alpha)\mathfrak{B}_1$ is contained in $\mathfrak{B} = \Sigma\rho(\sigma)\mathfrak{B}_0$, and we have $R \supseteq S^{\S} \supseteq S$, where $(S^{\S})_l = V_{\mathfrak{A}}(\mathfrak{B}^{\S})$. If here $R$ is regular with respect to $\mathfrak{B}_1$,[17] then it is regular with respect to $\mathfrak{B}^{\S}$ and $S^{\S}{}_l$, and we have $\mathfrak{B}^{\S} = V_{\mathfrak{A}}(S^{\S}{}_l)$. Now the following lemma may be applied to $\mathfrak{C}_0 = \mathfrak{B}_1$ provided its supplementary conditions are fulfilled: [18]

LEMMA 4. *Assume that $\mathfrak{B}_0 = R_rK_l$ has an independent basis over $R_r$ consisting of elements in $K_l$. Let $\mathfrak{C}_0$ be an $R_r$-two-sided submodule of $\mathfrak{B}_0$ such that $\mathfrak{C}_0N_r = \mathfrak{C}_0 \cap N_rK_l$, where $N$ denotes the radical of $R$, and that $\mathfrak{C}_0/\mathfrak{C}_0 \cap N_rK_l$ is $R_r/N_r$-right-regular.*[19] *Then $\mathfrak{C}_0$ is a direct summand of $\mathfrak{B}$ as $R_r$-two-sided module and has also an independent basis over $R_r$ consisting*

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[15] $R_r\Phi_0$ denotes (not the mere product but) the product-module.

[16] See [10], Hilfssatz 5. The assumption there may be weakened so as to include the case of rings which are not algebras.

[17] This is the case if $\mathfrak{B}_0$ is right-regular with respect to $\mathfrak{B}_1$.

[18] We shall indeed apply the lemma to $\mathfrak{U}_\sigma$, $\mathfrak{U}^*{}_\sigma$ in § 6, 7, which satisfy our additional requirements.

[19] Under the first assumption, $\mathfrak{C}_0 \cap N_rK_l = \mathfrak{C}_0N_r$, this is of course equivalent to the statement that $\mathfrak{C}_0/\mathfrak{C}_0N_r$ is $R_r/N_r$-right-regular, and this last is certainly the case when $\mathfrak{C}_0$ is $R_r$-right-regular.

*of elements of* $K_l$. (*If* $R$ *is primary, the condition* $\mathfrak{C}_0 N_r = \mathfrak{C}_0 \cap N_r K_l$ *is satisfied by every* $R_r$*-right-regular* $\mathfrak{C}_0$.)

This can be seen as in [9], § 3. For the sake of completeness we repeat the argument briefly. First $\mathfrak{C}_0 \bmod N_r K_l$ (i. e. $\bmod \mathfrak{C}_0 \cap N_r K_l$) is an $R_r/N_r$-two-sided submodule of $R_r K_l/N_r K_l$. This last residue-module possesses an independent basis over $R_r/N_r$ commutative with $R_r/N_r$ elementwise, and so is a direct sum of $R_r/N_r$-two-sided moduli isomorphic with simple components of $R_r/N_r$. Hence $\mathfrak{C}_0/\mathfrak{C}_0 \cap N_r K_l$ possesses a similar structure. But the numbers of its components isomorphic with simple components of $R_r/N_r$ are all equal, by virtue of its $R_r/N_r$-regularity. Hence $\mathfrak{C}_0/\mathfrak{C}_0 \cap N_r K_l$ is also decomposed into $R_r/N_r$-two-sided submoduli isomorphic to $R_r/N_r$ and so possesses an independent basis over $R_r/N_r$ commutative elementwise with $R_r/N_r$. Let the basis classes be represented by $\gamma_1, \gamma_2, \cdots, \gamma_t$ $(\varepsilon \mathfrak{C}_0)$; $\mathfrak{C}_0 \equiv \gamma_1 R_r + \gamma_2 R_r + \cdots + \gamma_t R_r \bmod N_r K_l$. Since, however, $\mathfrak{C}_0 \cap N_r K_l =$ (whence $\subseteq$) the intersection $\mathfrak{C}_0 N_r$ of all maximal $R_r$-right-submoduli by assumption, we have $\mathfrak{C}_0 = \gamma_1 R_r + \cdots + \gamma_t R_r$ too. (The sum is direct as we shall see in a moment.) Further, there exists a system $a_1, a_2, \cdots, a_s$ of elements in $K$ such that $\gamma_1, \cdots, \gamma_t, a_{1l}, \cdots, a_{sl} \bmod N_r K_l$ together form an independent $R_r/N_r$-basis of $R_r K_l/N_r K_l$. We see, by an argument similar to the above, that

$$\begin{aligned} R_r K_l &= \gamma_1 R_r + \cdots + \gamma_t R_r + a_{1l} R_r + \cdots + a_{sl} R_r \\ &= \mathfrak{C}_0 + a_{1l} R_r + \cdots + a_{sl} R_r. \end{aligned}$$

Here the sums are direct as can be seen by consideration of $R_r$-lengths. Thus $\mathfrak{C}_0$ is a direct summand in $R_r K_l$ as $R_r$-two-sided module, and so is a direct sum of submoduli ($R_r$-two-sided) isomorphic to directly indecomposable two-sided ideal components of $R_r$. On observing that $\mathfrak{C}_0/\mathfrak{C}_0 N_r$ is regular, we see then that $\mathfrak{C}_0$ is a direct sum of submoduli $R_r$-two-sided isomorphic to $R_r$, which proves the lemma.

**6.** We next start conversely with a ring $T$ between $R$ and $S$; $R \supseteq T \supseteq S$. We have

$$R_r = V_{\mathfrak{A}}(R_l) \subseteq V_{\mathfrak{A}}(T_l) \subseteq V_{\mathfrak{A}}(S_l) = \mathfrak{B} = \Sigma \rho(\sigma) \mathfrak{B}_0.$$

Hence $V_{\mathfrak{A}}(T_l) = \Sigma \rho(\sigma) \mathfrak{U}_\sigma$ for certain $R_r$-two-sided submoduli $\mathfrak{U}_\sigma$ of $\mathfrak{B}_0$; observe the Galois class-group property of the system $\{\rho(\sigma)\}$.[20] In particular $\mathfrak{U}_1 \supseteq R_r$.

[20] The same is the case for every $R_r$-two-sided submodule of $\mathfrak{B}$ and in particular for every subring of $\mathfrak{B}$ containing $R_r$.

Assume now that $R$ is ($T_l$-right-, that is) $T$-left-regular. Then $T$ is $S$-left-regular. Naturally $R_r$ is $T_r$-left-regular and $R_l$ is $T_l$-right-regular. Further, $V_{\mathfrak{R}}(T_l)$ is $R_r$ $(= V_{\mathfrak{R}}(R_l))$-right-regular. Hence each $\mathfrak{U}_\sigma$ is a direct sum of submoduli $R_r$-right-isomorphic to directly indecomposable right-ideal direct components of $R$. Concerning these $R_r$-right-components of $\mathfrak{U}_\sigma$ we have

LEMMA 5. *Under our assumption that $R$ is $T$-left-regular, no direct component of $\mathfrak{U}_\sigma$ as $R_r$-right-module is contained in $\mathfrak{B}_0 N_r$, whence $\mathfrak{U}_\sigma \cap \mathfrak{B}_0 N_r = \mathfrak{U}_\sigma N_r$, or, which is the same, $\mathfrak{U}_\sigma \cap \mathfrak{B}_0 N_r$ coincides with the intersection of all the maximal $R$-right-submoduli of $\mathfrak{U}_\sigma$, $N$ being the radical of $R$.*

To prove this, let [21] the $\mathfrak{B}_0$-right-rank of $R$ be $b = n/m$, with integral $n, m$. The $R_r$-right-rank of $\mathfrak{B}_0$ is then $b^{-1} = m/n$. Denote the $T$-left-rank of $R$ by $h = w/v$, with integers $w, v$. $T$ is $S$-left-regular with rank $b^{-1}gh^{-1} = mgv/nw$. Hence $V_{\mathfrak{R}}(S_l) = \mathfrak{B} = \Sigma\rho(\sigma)\mathfrak{B}_0$ is $V_{\mathfrak{R}}(T_l)$-right-regular with rank $mgv/nw$. We consider the direct sum of $niw$ copies of $\mathfrak{B}$:

$$\mathfrak{B}^{nw} = x^{(1)}\mathfrak{B} + x^{(2)}\mathfrak{B} + \cdots + x^{(nw)}\mathfrak{B}.$$

Let $R$ be a direct sum of $r$ directly indecomposable right-ideals (and left-ideals too). Our module $\mathfrak{B}^{nw}$ is a direct sum of $wmgr$ directly indecomposable $R_r$-right-moduli. We consider the module modulo the submodule $\mathfrak{B}^{nw}N_r = (\mathfrak{B}N_r)^{nw}$. The residue-module is decomposed into $wmgr$ irreducible $R_r$-right-moduli.

On the other hand, our module $\mathfrak{B}^{nw}$ is ($V_{\mathfrak{R}}(T_l)$-, whence) $R_r$-isomorphic to $V_{\mathfrak{R}}(T_l)^{mgv}$. So, if $\mathfrak{U}_\sigma$ is decomposed into $t_\sigma$ directly indecomposable $R_r$-right-moduli, then we must have $mgv\Sigma t_\sigma = wmgr$, or, $v\Sigma t_\sigma = wr$. We assert that none of these $t_\sigma$, for each direct component of $\mathfrak{U}_\sigma$, is contained in $\mathfrak{B}_0 N_r$. Indeed, if such a component, say $\mathfrak{u}_\sigma$, were contained in $\mathfrak{B}_0 N_r$, then $\rho(\sigma)\mathfrak{u}_\sigma \subseteq \mathfrak{B}N_r$, and $\mathfrak{B}^{nw}$ $(\simeq V_{\mathfrak{R}}(T_l)^{mgv})$ mod $\mathfrak{B}^{nw}N_r$ would be a direct sum of a smaller number of irreducible moduli than $mgv\Sigma t_\sigma$ $(= wmgr)$,[22] which is a contradictiou. Thus no direct ($R_r$-)component of $\mathfrak{U}_\sigma$ is contained in $\mathfrak{B}_0 N_r$, and the lemma is proved.

Our lemma holds without the assumption that $\mathfrak{B}_0$ has the form $R_r K_l$. If however it is assumed and if indeed $\mathfrak{B}_0$ has an independent $R_r$-basis

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[21] Cf. [8], § 2.

[22] Observe that each component contributes at most one irreducible component to the residue-module.

consisting of elements of $K_l$, then the lemma says that Lemma 4, of the preceding section, can be applied to our $\mathfrak{U}_\sigma$ in place of $\mathfrak{C}_0$.

**7.** Now let $\mathfrak{B}_0 = R_r K_l$ $(K_l = \mathfrak{B}_0 \cap R_l)$ be two-sided directly indecomposable. Let $\Phi$, $\Phi_0$ be the totalities of (ring-)automorphisms of $R$ contained in $\mathfrak{B}$, $\mathfrak{B}_0$, respectively, as in § **4**. Consider a ring $\mathfrak{C}$ between $\mathfrak{B}$ and $R_r$; $\mathfrak{B} \supseteq \mathfrak{C} \supseteq R_r$. $\mathfrak{C}$ has the form $\mathfrak{C} = \Sigma\rho(\sigma)\mathfrak{U}_\sigma$. Put $\mathfrak{C} \cap \Phi = \Psi$ and $\mathfrak{C} \cap \Phi_0 = \Psi_0$. These are groups.[23] Choose a second representative system $\{\rho^*(\sigma)\}$ of $\Phi$ mod $\Phi_0$ such that its subsystem, say $\{\rho^*(1), \rho^*(\alpha), \cdots, \rho^*(\beta)\}$, forms a representative system of $\Psi$ mod $\Psi_0$. Clearly, $\{1, \alpha, \cdots, \beta\}$ is a subgroup of $G$, which we shall denote by $H$. Also $\mathfrak{B} = \Sigma\rho^*(\sigma)\mathfrak{B}_0$, and $\mathfrak{C} = \Sigma\rho^*(\sigma)\mathfrak{U}^*_\sigma$ with submoduli $\mathfrak{U}^*_0$ of $\mathfrak{B}_0$; here $\mathfrak{U}^*_\alpha \supseteq R_r$ for $\alpha \in H$.

If in particular $\mathfrak{C} = V_{\mathfrak{A}}(T_l)$ with $R \supseteq T \supseteq S$, and $R$ is $T$-left-regular, then each $\mathfrak{U}^*_\sigma$ is a direct sum of $R_r$-right-moduli isomorphic with right-ideal direct components of $R$, and moreover $\mathfrak{U}^*_\sigma N_r = R_r K_l N_r \cap \mathfrak{U}^*_\sigma = N_r K_l \cap \mathfrak{U}^*_\sigma$, as was shown in the preceding section. If here $R$ is primary, $\mathfrak{U}^*_\sigma$ is $R_r$-right-regular unless it vanishes. If further $R_r K_l$ has an independent basis over $R_r$, consisting of elements of $K_l$, then, by Lemma 4, each $\mathfrak{U}^*_\sigma$ has also an independent $R_r$-basis $\{a_{il}{}^{(\sigma)}$ $(i = 1, 2, \cdots, u_\sigma)\}$ consisting of elements of $K_l$; $\mathfrak{U}^*_\sigma = \sum_{i=1}^{u_\sigma} R_r a_{il}{}^{(\sigma)}$ $(a_i{}^{(\sigma)} \in K)$. Supposing moreover that all the $a_i{}^{(\sigma)}$ (with varying $\sigma, i$) are regular elements of $K$ (which is certainly the case if $K$ mod $K \cap N$ is a division ring),[24] we assert

LEMMA 6. *Under all these assumptions,* $\mathfrak{C} = R_r\Psi = \sum_{\alpha \in H} \rho^*(\alpha) R_r \Psi_0$, *and in fact* $\mathfrak{U}^*_\sigma = R_r\Psi_0 = R_r L_l$ *whenever* $\mathfrak{U}^*_\sigma \neq 0$, *where* $L$ *is the module spanned by* $a_1{}^{(1)}, a_2{}^{(1)}, \cdots$ *over the center* $Z$ *of* $R$ *and satisfies* $L_l = R_l \cap \mathfrak{C} = R_l \cap \mathfrak{U}^*_\sigma$, $L = V_R(T)$.

For, on taking $\sigma$ with $\mathfrak{U}^*_\sigma \neq 0$, we have $\mathfrak{U}^*_\sigma \supseteq R_r a_{il}{}^{(\sigma)} \ni (a_i{}^{(\sigma)})_r{}^{-1} a_{il}{}^{(\sigma)}$ and $\rho^*(\sigma)(a_i{}^{(\sigma)})_r{}^{-1} a_{il}{}^{(\sigma)} \in \mathfrak{C} \cap \Phi = \Psi$. So $\sigma \in H$ and

$$(a_i{}^{(\sigma)})_r{}^{-1} a_{il}{}^{(\sigma)} = \rho^*(\sigma)^{-1}\rho^*(\sigma)\ (a_i{}^{(\sigma)})_r{}^{-1} a_{il}{}^{(\sigma)} \in \mathfrak{B}_0 \cap \Psi = \Psi.$$

Thus $R_r a_{il}{}^{(\sigma)} = R_r (a_i{}^{(\sigma)})_r{}^{-1} a_{il}{}^{(\sigma)} \subseteq R_r\Psi_0$ and $\mathfrak{U}^*_\sigma \subseteq R_r\Psi_0$. Therefore

$$\mathfrak{C} = \Sigma\rho^*(\sigma)\mathfrak{U}^*_\sigma \subseteq \sum_{\alpha \in H} \rho^*(\alpha) R_r \Psi_0.$$

Since the inclusion in the opposite direction is evident, we have

$$\mathfrak{C} = \sum_{\alpha \in H} \rho^*(\alpha) R_r \Psi_0 = R\Psi.$$

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[23] $\mathfrak{C}$ satisfies, together with $\mathfrak{B}$, $R_r$-(whence $\mathfrak{C}$-) chain condition.

[24] Since $a_i{}^{(\sigma)}$ are not contained in $N$ anyway; if $a \in N$, then $N_r{}^t a_l = 0$ with $N^t \neq 0$.

So $\mathfrak{U}^*_\sigma = R_r\Psi_0 = \mathfrak{U}^*_1$ whenever $\mathfrak{U}^*_\sigma \neq 0$. Thus $\mathfrak{U}^*_\sigma = R_rL_l$ (for such $\sigma$). Further the cited properties of $L$ are also evident.

Now we formulate our

THEOREM 2. *Let $R$ be a primary ring (with unit element and satisfying minimum condition). Let $\Phi$ be a group automorphism of $R$ which satisfies the following conditions*: i) *the subring $K$ of $R$ defined by $K_l = R_r\Phi \bigcap R_l$,*[25] *certainly containing the center $Z$ of $R$, is regular over $Z$, or, what amounts to the same (since $Z$ is completely primary), $K$ has an independent (finite) basis over $Z$,* ii) *$K \bmod K \bigcap N$ is a division ring, where $N$ denotes the radical of $R$,*[26] iii) *$R$ is $R_rK_l$-regular,* iv) *every automorphism of $R$ contained in $R_rK_l$ is contained in $\Phi$,* v) *the (necessarily) invariant subgroup $\Phi_0 = \Phi \bigcap R_rK_l$ has a finite index in $\Phi$, and* vi) *there exists a representative system $\{\rho_i\}$ of $\Phi \bmod \Phi_0$ such that $\rho_iR_r$, $\rho_jR_r$ with $i \neq j$ have no $R_r$-two-sided isomorphic composition residue-moduli.*[27, 28]

*Let $S$ be the invariant system of $\Phi$ in $R$, or, which is the same, the totality of elements of $R$ commutative elementwise with $K$ and invariant under $\{\rho_i\}$. Then $\Phi$ exhausts those automorphisms of $R$ which leave $S$ elementwise invariant. $R$ has an $S$-left-basis of $(\Phi : \Phi_0)(K : Z)$ terms.*

*Let $T$ be a ring between $R$ and $S$: $R \supseteq T \supseteq S$, such that $R$ is $T$-left-regular. Let $\Psi$ be the totality of automorphisms of $R$ leaving $T$ elementwise fixed. Then $T$ is the invariant system of $\Psi$ in $R$. $\Psi$ possesses a similar structure as $\Phi$ with a subring $L$ of $K$ in place of $K$, which satisfies the properties* i)-vi) *with $K, \Phi, \Phi_0$ replaced by $L, \Psi, \Psi_0 = \Psi \bigcap R_rL_l$ $(= \Psi \bigcap R_rR_l)$. Further, not only is $R$ $T$-left-regular but $R$ has an independent $T$-left-basis of $(\Psi : \Psi_0)(L : Z)$ terms, and $T$ has an $S$-left-basis of $(\Phi : \Phi_0)(K : Z)/(\Psi : \Psi_0)(L : Z)$ terms.*

*If conversely $\Psi$ is a subgroup of $\Phi$ such that $L$, defined by $L_l = R_r\Psi \bigcap R_l$, has an independent basis over $Z$,*[29] *and every automorphism of $R$ contained*

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[25] See footnote 15.

[26] ($K$ is then completely primary, whence directly indecomposable, and therefore) $R_rK_l$ is (two-sided) directly indecomposable, since every direct summand of $R_rK_l$ as $R_r$-two-sided module possesses an $R_r$-basis consisting of elements of $K_l$; of course i) being taken into account.

[27] That is, $R$-two-sided moduli $(\rho_i, R)$, $(\rho_j, R)$ with $i \neq j$, in the notation of § 0, have no isomorphic composition residue-moduli.

[28] Then $\Phi_0 = \Phi \bigcap R_rR_l$ too.

[29] Then we see that $K$ has a right-, say, basis over $L$ by an argument similar to one in our proof of Lemma 4, observing that $L \bmod N$ is a division ring. So ($R_rK_l$ whence) $R$ is $R_rL_l$-regular. We may also replace conversely our assumption by the $L$-right-, say, regularity of $K$.

*in $R_rL_l$ is in $\Psi$ (i. e. the properties* i), iv) *with $\Psi, L$ in place of $\Phi, K$ are satisfied),*[30] *then $R$ has an independent left-basis over the invariant system $T$ of $\Psi$ (whence $R$ is certainly $T$-left-regular), and $\Psi$ exhausts the automorphisms of $R$ leaving $T$ elementwise invariant.*

*Thus such between-rings $T$ (over which $R$ is left-regular) and such subgroups $\Psi$ (having the described structure) are in* 1-1 *dual correspondence.*

Proof is immediate by virtue of our preparations.

**8.** Our Theorem 2 includes (the main) Theorem 8 of [5] (and the corresponding statement in [2]) as a special case where $R$ is a division ring, if the following fact is taken into consideration:

LEMMA 7.[31] *If $R$ is a division ring and $K$ a subring containing the center $Z$ of $R$, then every automorphism of $R$, whose inverse is contained in $R_rK_l$ is an inner automorphism generated by an element of $K$.*

To see this, first let an automorphism $\phi$ of $R$ be contained in $R_rR_l$; $\phi - (t_{1r}s_{1l} + t_{2r}s_{2l} + \cdots) = 0$, where $t, s$ $(\neq 0) \in R$ and $t_{1r}, t_{2r}, \cdots$ are linearly independent over $R_l$ (i. e. $t_1, t_2, \cdots$ are so over $Z$). For $x \in R$, $0 = x_l\phi - x_l(t_{1r}s_{1l} + \cdots) - \phi x_l{}^{\phi} + (t_{1r}s_{1l} + \cdots)x_l{}^{\phi} = t_{1r}(s_{1l}x_l{}^{\phi} - x_ls_{1l}) + \cdots$. Hence $s_{1l}x_l{}^{\phi} - x_ls_{1l} = 0$, that is, $x^{\phi}s_1 - s_1x = 0$, $x^{\phi} = s_1xs_1{}^{-1}$, and $\phi$ is the inner automorphism $s_{1r}{}^{-1}s_{1l}$, generated by $s_1{}^{-1}$. Suppose now $\phi \in R_rK_l$. Then $s_{1l} \in s_{1r}R_rK_l = R_rK_l$, whence $\in R_rK_l \cap R_l = K_l$, i. e. $s_1 \in K$, which proves our assertion.

We note that the somewhat complicated, though not irrational, assumptions in our Theorem, concerning $K$, become rather natural (and in fact some of them are satisfied automatically) also in case $R$ is a simple ring,[32] and similarly in the case when $K$, instead of $R$, is simple.[33]

On the other hand, it is very desirable to replace the assumption of primarity[34] of $R$ by mere direct indecomposability. Indeed we have done that in case $\Phi = 1$. Namely: *Let $\mathfrak{G}$ be a Galois group*[35] *of a ring $R$ (with*

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[30] Besides ii), which is trivial, and iii), which was observed in the preceding footnote, properties v), vi) with $\Psi$, $L$ in place of $\Phi$, $K$ are satisfied automatically. Cf. footnote 28.

[31] Cf. [5], Lemma 2, which in turn is contained in our argument of the Galois class-group and idealistic components.

[32] Cf. a remark at the end of [9], § 3.

[33] See [9], § 2, Example.

[34] As a matter of fact, in case of a primary ring $R$ our Lemma 5 was superfluous, or evident from the regularity of $\mathfrak{U}_{\sigma}$, $\mathfrak{U}^*_{\sigma}$.

[35] That is, a finite group of automorphisms which forms by itself a representative system of a Galois class-group. See [8].

*unit element and satisfying minimum condition*), *and let* $S$ *be the invariant system of* $\mathfrak{G}$ *in* $R$. (*Then* $R$ *is both left- and right-regular with rank* $g=(\mathfrak{G}:1)$ *over the invariant system* $S$ *of* $\mathfrak{G}$. *If* $R$ *is two-sided directly indecomposable,* $\mathfrak{G}$ *exhausts automorphisms of* $R$ *leaving* $S$ *elementwise invariant.*) *Under the same assumption of direct indecomposability of* $R$, *subgroups of* $\mathfrak{G}$ *are in* 1-1 *dual correspondence, in the usual manner of the Galois theory, with subrings of* $R$, *containing* $S$, *over which* $R$ *is right-* (*or left-*)*regular.* The only point which is needed to establish this and which is not covered by our argument in the present note is that of showing $\mathfrak{U}_\sigma = R_r$ for $\mathfrak{U}_\sigma \neq 0$ (when $\mathfrak{B}_0 = R_r$) (see § **6** above). We want to refer for it to the latter half of the proof given to Theorem 2 in [8].

The writer feels that the strict division into commuter and Galois class-group parts, so to speak, is not very natural in case $R$ is not a division ring, or at least a simple ring. In all, he does not assume that the above is the final form of the theory; he hopes to come back to the problem elsewhere. A non-normal theory, which generalizes [4], will be given in a subsequent paper.

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# LINEARLY COMPACT SPACES AND DOUBLE VECTOR SPACES OVER SFIELDS.*

By J. DIEUDONNÉ.

**Introduction.** The notion of *linearly compact vector space* has recently been introduced by S. Lefschetz, in view of applications to Algebraic Topology ([7] p. 78). It is closely connected with the general theory of *weak duality* of vector spaces over topological sfields, which I have developed for algebraic purposes [2]; and I shall in fact show in Section 1 how the use of that theory greatly simplifies the proof of Lefschetz's main structure theorem. In the remainder of the paper, I will use linearly compact vector spaces to complete the recent generalization of Jacobson's theory of "composites" of self-representations of fields [5] developed by G. Hochschild [4]; it turns out that they are the proper tool by which Hochschild's finiteness restrictions may be entirely dropped.

**1. Linearly compact spaces.** Let $K$ be an arbitrary sfield, on which the topology will always be the discrete topology. A right topological vector space $E$ over $K$ is said to be *linearly compact* when:

a) it is a Hausdorff space in which there is a fundamental system of neighborhoods of 0 consisting of linear subspaces of $E$;

b) any filter base on $E$ consisting of closed linear varieties has a non-empty intersection.

Spaces satisfying a) are said to be *linearly topologized.* The following are easy consequences of the definition ([7], p. 78):

c) a product of linearly compact spaces is linearly compact;

d) if $E$ is linearly compact, and $H$ is a closed linear subspace of $E$, then $H$ and $E/H$ are linearly compact;

e) let $E$ be linearly compact, $F$ linearly topologized, and let $u$ be a continuous linear mapping of $E$ into $F$; then $u(E)$ is linearly compact, and $u$ is a homomorphism [1] of $E$ onto $u(E)$;

f) a discrete linearly compact space is finite dimensional.

Let $E$ be a linearly compact space, $E'$ its *dual* in the *topological* sense, namely the left vector space (over $K$) consisting of all *continuous* linear

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[1] We use the term "homomorphism" in the sense of Bourbaki, meaning what is called "open homomorphism" by some authors.

mappings of $E$ into $K$ (continuous linear forms on $E$). Assumption a) implies that for every $x_0 \neq 0$ in $E$ there exists a closed hyperplane $H$ such that $x_0 \notin H$; this amounts to saying that there exists a continuous linear form $x' \in E'$ such that $x'(x_0) \neq 0$; $E$ may therefore be considered as a subspace of the *dual* $E'^*$ of $E'$ in the *algebraic* sense, that is, the right vector space of *all* linear forms on $E'$. Now we prove the following properties:

g) *The topology on $E$ is identical to the weak topology $\sigma(E, E')$* (topology of pointwise convergence in $E'$). Indeed, let $V$ be an open (and closed) linear subspace of $E$; $E/V$ is linearly compact by d), and discrete, therefore finite dimensional by f); hence $V$ may be defined by a finite number of equations $x'_i(x) = 0$, with $x'_i \in E'$, and is therefore a neighborhood of 0 for $\sigma(E, E')$, which proves our contention.

h) *The linearly compact space $E$ is complete.* Let $\mathfrak{F}$ be a Cauchy filter on $E$; for every open linear subspace $V$ of $E$ and every set $A \in \mathfrak{F}$ let us consider the set $A + V$; it is readily seen that these sets constitute a Cauchy filter. In fact, for every such open subspace $V$, there exists $A \in \mathfrak{F}$ such that for $x \in A$ and $y \in A$, $x - y \in V$, hence $A + V = x + V$, $A + V$ is a closed linear variety. Clearly the sets $A + V$ which are linear varieties form a filter base, hence have by assumption a common point $x_0$, which is the limit of the filter $\mathfrak{F}$.

The main structure theorem of Lefschetz is now an immediate consequence of g) and h) and of the well-known fact that $E$, with the topology $\sigma(E, E')$ may only be complete when $E = E'^*$ ([2], p. 66); $E$ is thus the *algebraic dual* of a left vector space $E'$ over $K$; if $(e_\lambda)_{\lambda \in L}$ is a basis for $E'$, $E$ is thus algebraically isomorphic to the *product vector space* $K^L$ and the topology $\sigma(E, E')$ on $E$ is identical to the product topology; this is Lefschetz's theorem ([7], p. 83, theorem (32.1)).

On the dual $E'$ of $E$, Lefschetz considers the discrete topology (which is also the topology of uniform convergence in $E$); it will be more useful for us to consider the *weak topology* $\sigma(E', E)$ (topology of pointwise convergence in $E$); we stress that for that topology as well as for the discrete topology, *every* linear subspace in $E'$ is *closed*.

**2. Cyclic double modules and relations spaces.** If $A$ and $B$ are two rings with identity elements, we consider $(A, B)$-modules, that is, sets $M$ having simultaneously the structure of a left $A$-module and of a right $B$-module, with the conditions $1 \cdot x = x \cdot 1 = x$ and $(\alpha x)\beta = \alpha(x\beta)$ for $x \in M$, $\alpha \in A$, $\beta \in B$; such a module $M$ is called *cyclic* if there exists an element $m \in M$ such that $M = A \cdot m \cdot B$ (set of all linear combinations $\sum_i \alpha_i m \beta_i$); $m$ is then called a generator of $M$.

We shall be concerned with cyclic $(K, A)$-modules, where $K$ is a *sfield* and $A$ a ring with identity. Let $M$ be such a module, $m_0$ a generator of $M$; $M$ is a left vector space over $K$. We consider with Hochschild [4], the algebraic *dual* $M'$ of that vector space; this is a right vector space over $K$, $x'\lambda$ being the linear form $x \to x'(x)\lambda$. On the other hand it may be considered as a left $A$-module, the product $u \cdot x'$ of $u \in A$ and $x' \in M'$ being the linear form $x \to x'(x \cdot u)$. The usual axioms are readily verified, as well as the condition $u \cdot (x'\lambda) = (u \cdot x')\lambda$ for $\lambda \in K$ and $u \in A$, and therefore $M'$ is an $(A, K)$-module.

Next we associate to $M$ and $m_0$ what Hochschild calls the *relations space* of $M$ (relative to $m_0$). Let $\mathcal{E}$ be the additive group of *all* representations of the additive group $A$ into the additive group $K$; to every element $x' \in M'$ we associate the element $\xi = \phi(x')$ of $\mathcal{E}$ defined by $\xi(u) = x'(m_0 \cdot u)$ for every $u \in A$; owing to the fact that $m_0$ is a generator of $M$, $\phi$ is a one-one mapping of $M'$ onto a subgroup $R(M, m_0)$ of $\mathcal{E}$. Moreover, one may define on $\mathcal{E}$ a structure of $(A, K)$-module, by setting $\xi\lambda$ $(\lambda \in K)$ equal to the representation $w \to \xi(u)\lambda$, and $v \cdot \xi$ $(v \in A)$ to the representation $u \to \xi(u \cdot v)$; it is then clear that $R(M, m_0)$ is an $(A, K)$-submodule of $\mathcal{E}$, and $\phi$ an $(A, K)$-isomorphism of the $(A, K)$-module $M'$ onto $R(M, m_0)$.

Now consider on $M'$ the weak topology $\sigma(M', M)$, and on $\mathcal{E}$ the topology of pointwise convergence in $A$; for these topologies, $\phi$ is a *homeomorphism* of $M'$ onto $R(M, m_0)$. In fact, every finite subset $(x_i)_{1 \leq i \leq n}$ of $M$ is contained in a subspace (over $K$) generated by a finite number of elements of the form $m_0 \cdot u_j$ $(1 \leqq j \leqq p)$, and therefore the relations $x'(x_i) = 0$ (which define a neighborhood of 0 in $M'$) are implied by $x'(m_0 \cdot u_j) = 0$, that is $(\phi(x'))(u_j) = 0$; this proves that $\phi$ maps a neighborhood of 0 in $M'$ onto a neighborhood of 0 in $R(M, m_0)$; the same property for $\phi^{-1}$ is obvious.

The characterization of relations spaces $R(M, m_0)$ in $\mathcal{E}$, generalizing Hochschild's Theorem 5.3 (which concerns the finite case) now reads as follows:

THEOREM 1. *In order that an $(A, K)$-submodule $T$ of $\mathcal{E}$ be the relations space of a cyclic $(K, A)$-module, it is necessary and sufficient that $T$, with the topology induced by that of $\mathcal{E}$, be a linearly compact right vector space over $K$.*

We observe first that an $(A, K)$-submodule $T$ of $\mathcal{E}$ is always linearly topologized by the topology induced by that of $\mathcal{E}$, for a neighborhood $V$ of 0 consists of all $\xi \in T$ such that $\xi(u_i) = 0$ for a finite number of $u_i \in A$, and it is clear that the relation $\xi \in V$ implies $\xi\lambda \in V$ for every $\lambda \in K$, which proves that $V$ is a linear subspace of $T$.

The necessity of the condition on $T$ is a consequence of the remarks preceding Theorem 1. To prove the sufficiency, we proceed as in the proof of Hochschild's Theorem 5.3, and consider the *dual* $M = T'$ of the right vector space $T$ (in the topological sense). $M$ may be made into a $(K, A)$-module by setting $\lambda m$ $(\lambda \in K)$ equal to the mapping $\xi \to \lambda m(\xi)$ (this is the usual structure of left vector space of $T'$), and $m \cdot u$ $(u \in A)$ equal to the mapping $\xi \to m(u \cdot \xi)$. Let $m_0$ be the mapping $\xi \to \xi(1)$ of $T$ into $K$, which is obviously a continuous linear form, and therefore belongs to $M$; we are going to prove that $M = K \cdot m_0 \cdot A$. It is clear that $K \cdot m_0 \cdot A$ is a linear subspace of the vector space $T' = M$; as all linear subspaces of $T'$ are closed for the topology $\sigma(T', T)$, it will be enough to show that $K \cdot m_0 \cdot A$ is *dense* in $T'$ for that topology. We use the following lemma of Hochschild ([4], Lemma 2.1):

LEMMA 1. *Let $S$ be an arbitrary set, $V$ a linear subspace of finite dimension $n$ in the right $K$-vector space $K^S$ of all mappings of $S$ into the sfield $K$; then there exist $n$ elements $s_i$ $(1 \leqq i \leqq n)$ of $S$ and a basis $(\xi_i)_{1 \leq i \leq n}$ of $V$ such that $\xi_i(s_j) = \delta_{ij}$.*[2]

Let then $\xi_i$ $(1 \leqq i \leqq n)$ be $n$ arbitrary elements of $T$, and $V$ the linear subspace of $T$ they generate; as $V \subset T \subset K^A$, there are $p$ elements $u_j$ $(1 \leqq j \leqq p)$ of $A$ and a basis $(\eta_j)_{1 \leq j \leq p}$ of $V$ such that $\eta_j(u_k) = \delta_{kj}$. If $m$ is an arbitrary element of $M$, and $m' = \sum_{j=1}^{p} m(\eta_j) \cdot m_0 \cdot u_j$, we have $m'(\eta_k) = \sum_{j=1}^{p} m(\eta_j) m_0(u_j \cdot \eta_k)$, and $m_0(u_j \cdot \eta_k) = (u_j \cdot \eta_k)(1) = \eta_k(u_j) = \delta_{jk}$, hence $m'(\eta_k) = m(\eta_k)$ for $1 \leqq k \leqq p$; accordingly $m'(\xi_i) = m(\xi_i)$ for $1 \leqq i \leqq n$, which proves that $K \cdot m_0 \cdot A$ is dense in $M$, and therefore equal to $M$. Finally, $T$ is the algebraic dual of $M$, when one sets $\xi(m) = m(\xi)$, and it follows from the definition of $m_0$ that $\xi(m_0 \cdot u) = \xi(u)$, hence the mapping $\phi$ is the identity, and we have proved that $T = R(M, m_0)$.

As a corollary to Theorem 1, we may prove Hochschild's theorems 6.2 and 6.3 *without any finiteness restriction.* We have first to show that, if $M$ and $N$ are cyclic $(K, A)$-modules with generators $m_0$ and $n_0$, then $R(M, m_0) + R(N, n_0)$ is the relations space of another cyclic $(K, A)$-module; it is enough to show, by Theorem 1, that if $T$ and $U$ are two linearly compact $(A, K)$-submodules of $\mathcal{E}$ then $T + U$ is also linearly compact. Now if we

[2] This may easily be deduced from the general theory of duality in vector spaces in the following way: for each $s \in S$, the mapping $\xi \to \xi(s)$ of $V$ into $K$ is a linear form $\bar{s}$ over $V$, and therefore $s \to \bar{s}$ is a mapping of $S$ into the dual $V'$ of $V$; moreover the image of $S$ by that mapping generates $V'$, for the relation $\xi(s) = 0$ for every $s \in S$ implies $\xi = 0$ in $V$. Therefore, there exist $n$ elements $s_i$ $(1 \leqq i \leqq n)$ such that the $\bar{s}_i$ constitute a basis of $V'$; the $\xi_i$ are then the elements of the "dual basis" of $V$ (see [1], p. 46).

consider the cartesian product $T \times U$, which is linearly compact by property c) of Section 1, the mapping $(\xi, \eta) \to \xi + \eta$ of $T \times U$ onto the subspace $T + U$ of $\mathcal{E}$ is linear and continuous; therefore $T + U$ is linearly compact by property e) of Section 1.

Next, when $M$ is decomposed into the direct sum of a finite number of cyclic $(K, A)$-modules $M_i$ $(1 \leqq i \leqq s)$, the dual $M'$ is a *direct topological sum* of the duals $M'_i$ (isomorphic to the cartesian product of the $M'_i$); these are of course the only kinds of decomposition of the relations space $R(M, m_0)$ which must be considered. With this condition on the decompositions of $R(M, m_0)$, Hochschild's Theorem 6.3 and its proof extend at once to the general case.

**3. Tensor products of cyclic double modules.** Let $H, K$ be two sfields, $A$ a ring with identity element. From a cyclic $(H, K)$-module $M$ with generator $m_0$, and a cyclic $(K, A)$-module $N$ with generator $n_0$, we construct a cyclic $(H, A)$-module $L$ with generator $m_0 \otimes n_0$ as in Hochschild ([4], § 7). Theorem 7.1 of Hochschild's paper may then be generalized as follows:

THEOREM 2. *The product* $R(M, m_0)R(N, n_0)$ (that is, the $(A, H)$-module generated by the elements $\xi \circ \eta$, where $\xi \in R(M, m_0)$ and $\eta \in R(N, n_0)$) *is dense in the relations space* $R(L, m_0 \otimes n_0)$.

We follow again closely Hochschild's proof. If $M'$ and $N'$ are the duals of $M$ and $N$, we define for every $x' \in M'$ and $y' \in N'$, the linear form $x'y'$ over $M \otimes N$ (hence over $L$) by the relation $x'y'(m \otimes n) = x'(m \cdot y'(n))$; then it is readily seen that $\phi(x'y') = \phi(x') \circ \phi(y')$, and therefore $R(M, m_0)R(N, n_0)$ is contained in $R(L, m_0 \otimes n_0)$. The remainder of Hochschild's proof may be simplified by the following argument: as $L$ is a linear subspace of the tensor product $M \otimes N$ (considered as a left vector space over $H$), every linear form over $L$ may be extended to a linear form over $M \otimes N$; our proof will then be complete if we show that, given any finite number of elements $z_k$ of $M \otimes N$, and an arbitrary linear form $z'$ over $M \otimes N$, there is a sum of linear forms of the type $x'y'$ which takes the same value as $z'$ at every $z_k$. We may suppose that the $z_k$ are of the form $m_i \otimes n_j$, where the $m_i$ are linearly independent in $M$ (over $H$) and the $n_j$ linearly independent in $N$ (over $K$); for a finite subset of $M \otimes N$ is always contained in the linear subspace spanned by a finite number of such elements. Now let $x'_i$ be a linear form over $M$ such that $x'_i(m_h) = \delta_{ih}$, and let $y'_j$ be a linear form over $N$ such that $y'_j(n_k) = \delta_{jk}$; one has then $x'_i y'_j(m_h \otimes n_k) = \delta_{ih}\delta_{jk}$, and therefore the linear combination $\sum_{i,j} \lambda_{ij} x'_i y'_j$, where the $\lambda_{ij}$ are arbitrary in $H$, takes the value $\lambda_{ij}$ at every element $m_i \otimes n_j$, and may therefore have at those elements the same value as an arbitrary linear form over $M \otimes N$.

Let us now consider only the case when $H$ is a subsfield of $K$ and $A = K$; we then define as Hochschild ([4], Definition 7. 2) the product of a cyclic $(H, K)$-module $M$ by itself, and say that $M$ is (algebraically) closed if it covers its product by itself. Hochschild's Theorem 7. 2 is then valid *without any finiteness assumption* on $M$: for if $R(M, m_0)$ is a ring, and therefore contains $R(M, m_0)R(M, m_0)$, it contains also the closure of that product, hence $M$ covers its product by itself; the converse is obvious.

To go further, we shall restrict ourselves to the case in which $H = K$. We need first the generalization of the Jacobson-Bourbaki theorem which was conjectured in [3], p. 163. The set $\mathcal{E}$ of all representations of the additive group $K$ into itself is then a *ring* and we prove the following:

THEOREM 3. *Let $T$ be a closed subring of $\mathcal{E}$, which at the same time is a $(K, K)$-submodule of $\mathcal{E}$. Let $L$ be the subsfield of $K$ consisting of all elements $u \in K$ such that $\xi(ux) = u\xi(x)$ for every $x \in K$ and every $\xi \in T$. Then $T$ coincides with the ring of all $L$-linear mappings of $K$ into itself.*[3]

As $T$ is closed, all we have to prove is that $T$ is *dense* in the ring $A$ of all $L$-linear mappings of $K$ into itself. This amounts to the following: let $b_i$ $(1 \leqq i \leqq n)$ be any finite number of elements of $K$, linearly independent over $L$; we must show that the $n$ elements $\xi(b_i)$ may assume *any* system of $n$ values in $K$, for a suitable $\xi \in T$; in other words, we must prove that the linear mapping $\xi \to (\xi(b_i))_{1 \leq i \leq n}$ of the right vector space $T$ (over $K$) into the right vector space $K^n$ is a mapping *onto* $K^n$. Now this is exactly what Bourbaki's argument (in [1], p. 72) proves: for it shows that in the opposite case there would exist a system $(\lambda_i)_{1 \leq i \leq n}$ of $n$ elements of $L$, not all 0, such that $\sum_{i=1}^{n} \lambda_i \xi(b_i) = 0$ for every $\xi \in T$, that is, $\xi(\sum_{i=1}^{n} \lambda_i b_i) = 0$ for all $\xi \in T$; but when $\xi(x) = 0$ for every $\xi \in T$, one has $x = 0$, for if $x \neq 0$ and $\xi \neq 0$, there exists $y \in K$ such that $\xi(y) \neq 0$, and that relation may be written $\eta(x) \neq 0$ with $\eta = (x^{-1}y) \cdot \xi \in T$. Therefore one would have $\sum_{i=1}^{n} \lambda_i b_i = 0$ against the hypothesis.

We recall that conversely, when $L$ is any subsfield of $K$, and $T$ is the (closed) ring of all $L$-linear mappings of $K$ into itself then $L$ is the set of all $u \in K$ such that $\xi(ux) = u\xi(x)$ for $x \in K$ and $\xi \in T$ ([6], p. 233, Theorem

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[3] (*Added in the proofs*). Professors N. Jacobson and H. Cartan have kindly informed me that they have been aware of this result for some time. Cartan's proof is identical with mine. Jacobson's proof runs as follows: $T$ is a right $K$-module, hence an *irreducible* ring of endomorphisms of $K$. It is therefore a *dense* ring of $L$-linear mappings of $K$ into itself ([6], p. 232, th. 6); as it is closed, it consists of *all* such mappings.

7). There exists therefore a 1-1 correspondence between subsfields of $K$ and closed subrings of $\mathcal{E}$ which are at the same time $(K, K)$-submodules.[4]

We may now give the characterization of all closed cyclic $(K, K)$-modules:

THEOREM 4. *If $K$ is any sfield, every closed cyclic $(K, K)$-module is equivalent with a tensor product $K \otimes_L K$ of $K$ with itself, with respect to a subsfield $L$ of $K$. Any two such products $K \otimes_L K$, $K \otimes_H K$ are equivalent if and only if $H = L$.*

Let $L$ be any subsfield of $K$, and $(e_\alpha)$ a basis of $K$, considered as a left vector space over $L$; then the elements $1 \otimes e_\alpha$ constitute a basis of $M = K \otimes_L K$ as a left vector space over $K$. For every element $\xi$ of the relations space $R(M, 1 \otimes 1)$ and every $\lambda \in L$ one has $\xi(\lambda u) = x'(1 \otimes (\lambda u)) = x'(\lambda(1 \otimes u)) = \lambda x'(1 \otimes u) = \lambda \xi(u)$ for every $u \in K$; this proves at once that $R(M, 1 \otimes 1)$ is identical with the ring of *all* $L$-linear mappings of $K$ into itself. In particular, the ring $\mathcal{E}$ is identical with the relations space of the $(K, K)$-module $K \otimes_P K$, where $P$ is the *prime field* contained in $K$; therefore $\mathcal{E}$, considered as a right vector space over $K$, is *linearly compact*; hence every *closed* linear subspace of $\mathcal{E}$ is also linearly compact (Section 1, property d)). Theorem 4 is then an immediate consequence of Theorems 1 and 3, and of Hochschild's Theorem 5. 2.

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[4] Instead of that condition, one may only suppose that the closed subrings considered are right vector spaces over $K$, and that for every $u \neq 0$ in $K$, there exists $\xi \in T$ such that $\xi(u) \neq 0$; both conditions imply that $T$ contains the subsfield of $\mathcal{E}$ consisting of all right multiplications $x \to xu$ by elements of $K$ (which may be identified with $K$), and conversely they are implied by that assumption.

# LOCALLY COMPACT RINGS. II.*

By IRVING KAPLANSKY.**

The purpose of this note is to complete the theory of the radical of a locally compact ring that was initiated in [4], and to prove two decomposition theorems illustrating the role of local direct sums in the theory.

THEOREM 1. *The radical of a locally compact ring is closed; more precisely, it is the intersection of the closed regular maximal right (or left) ideals.*

*Proof.* Let $A$ be a locally compact ring, $R$ its radical, $R'$ the closure of the latter. Then $R'$ is again a locally compact ring, and by [4, Lemma 2] its radical is $R$. If $R \neq R'$ then [4, Th. 11] $R'$ contains a closed regular maximal right ideal $M$. Then $R \subset M$, which contradicts the fact that $R$ is dense in $R'$.

We turn to the second statement of the theorem. Now that we know that the radical is closed, it will evidently suffice to treat the semi-simple case. Then by applying [4, Th. 2] we can reduce further to the case where $A$ is totally disconnected. Let $B$ be a compact open subring [4, Lemma 4], $S$ the radical of $B$, and $e$ an idempotent in $B$ mapping into the unit element of $B/S$ [3, Th. 16 and Lemma 12]. We shall now prove the following statement: any regular maximal right ideal $M$ in $A$ containing $eA$ is closed. Suppose on the contrary that $M$ is dense in $A$, and let $f$ be a left unit element modulo $M$. Then $M$ contains elements $f + x$ with $x$ arbitrarily small, say so small that $r = x - ex$ is in $S$ and is consequently quasi-regular: $r + s + rs = 0$. Now $ex \in M$ and so $f + r \in M$, $fs + rs \in M$. Since $f$ is a left unit modulo $M$, $fs - s \in M$. Hence $s + rs \in M$, $r \in M$, and $f \in M$, a contradiction.

Let now $y$ be an element lying in every closed regular maximal right ideal of $A$. For every idempotent $g$ of finite rank in $B$, we have $gy = 0$ by [4, Lemma 8]. Since $e$ is a limit of such idempotents, $ey = 0$. Consider the ideal $I = eA + (1 + y - ye)A$ (this is well-defined even if there is no

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unit element). If $I \neq A$, it can be expanded to a right ideal $M$ maximal with respect to exclusion of $y - ye$. Then $M$ will be actually maximal, regular since $ye - y$ is a left unit, and closed by the preceding paragraph. This means that $y$ is in $M$, a contradiction. It must therefore be the case that $I$ is all of $A$. Write

$$eb + (1 + y - ye)c = ye - e - y.$$

After left-multiplication by $1 - e$ this becomes $(y - ye) \circ (c - ec) = 0$. Thus $y - ye = y(1 - e)$ is right quasi-regular and so is $(1 - e)y = y$, as follows from the last identity on p. 154 of [3]—the lack of a unit element does not matter. Hence the intersection of the closed regular maximal right ideals consists entirely of right quasi-regular elements, and is therefore 0. Of course, by symmetry the same is true for left ideals.

An immediate corollary of Theorem 1 is the fact that a locally compact semi-simple ring is a subdirect sum of locally compact primitive rings. In the study of the kind of subdirect sum that may arise, it seems certain that a leading role will be played by the local direct sum introduced by Braconnier [2].

*Definition.* Let $A_i$ be topological rings with open subrings $B_i$. The *local direct sum* $A$ of $A_i$ relative to $B_i$ is the set of $\{a_i\}$, where $a_i \in A_i$ and all but a finite number are in $B_i$; the subring where all coordinates are in $B_i$ is given the Cartesian product topology and its neighborhoods of 0 are declared to be neighborhoods of 0 in $A$.

It is readily verified that this definition makes $A$ a topological ring; moreover $A$ is locally compact if and only if each $A_i$ is locally compact and all but a finite number of $B$'s are compact.

The next theorem shows that in a slight generalization of the commutative case a complete reduction to $Q$-rings is possible.

THEOREM 2. *A locally compact ring whose idempotents are in the center is a local direct sum of Q-rings.*

*Proof.* We begin by reducing the problem to the totally disconnected case. Let $A$ be the given ring, $C$ its component of 0. In $C$ we know [4, p. 448] that there is an ideal $P$ such that $P^2 = 0$ and $C/P$ is an algebra of finite order over the real numbers. The latter is the direct sum of an algebra with unit element $f$ and a nilpotent algebra. Let $g \in C$ be an idempotent mapping on $f$ modulo $P$. We take the Peirce decomposition

of $A$ relative to $g$, and we may confine further study to the component $B = A(1-g)$. In $B$ the component $D$ of 0 is nilpotent. Let $E$ be a compact open subring of the totally disconnected ring $B/D$, and choose an idempotent $h$ mapping on the unit element of $E$ modulo its radical. We further pick an idempotent $e$ in $B$ mapping on $h$ modulo $D$. In the Peirce decomposition of $B$ relative to $e$, the component $B(1-e)$ is a $Q$-ring. It remains to study the totally disconnected ring $Be$.

Let us start with $A$ totally disconnected, $F$ a compact open subring, and $e$, $e_i$ idempotents mapping respectively on the unit element and the primitive idempotents of $F$ modulo its radical. After a Peirce decomposition we may assume that $e$ is the unit element of $A$. We consider the mapping $a \to \{ae_i\}$, which is at least an algebraic isomorphism. The compact ring $F$ is the full Cartesian direct sum of $\{Fe_i\}$, as follows from [3, Th. 17]. Moreover a sequence with all but a finite number of its coordinates in $Fe_i$ will arise from a suitable element of $A$. Thus at least the local direct sum occurs (in its correct topology). Our theorem will be proved as soon as we verify that for each $a \in A$, all but a finite number of $ae_i$ are in $Fe_i$. There is a neighborhood $U$ of 0 such that $aU \subset F$. We may suppose that $U \subset F$, and such a neighborhood of $F$ consists of elements restricted at only a finite number of coordinates. At all the remaining coordinates $ae_i$ must be in $Fe_i$. This completes the proof of Theorem 2.

Let us now consider a locally compact totally disconnected $Q$-ring $A$. Then any compact open subring of $A$ has an open radical. In a context where all subrings of $A$ are semi-simple we can conclude that $A$ is discrete. All subrings of $A$ are semi-simple for example if $A$ is a Boolean ring, or more generally if every element satisfies $a^{n(a)} = a, n(a) > 1$. Combining this remark with Theorem 2, we see that a locally compact ring satisfying $a^{n(a)} = a$ is a local direct sum of discrete rings. We shall now prove a more general result by classifying all strongly regular locally compact rings. As defined in [1], a ring is strongly regular if for every $a$ there exists an $x$ with $a^2x = a$; we shall make free use of the results in [1].

THEOREM 3. *A strongly regular locally compact ring $A$ is of the form $X \oplus Y \oplus Z$ where $X$ is discrete, $Y$ is the direct sum of a finite number of non-discrete division rings, and $Z$ is a local direct sum of discrete strongly regular rings relative to finite subfields.*

*Proof.* By [4, Th. 2] we can confine ourselves to the totally disconnected case, and we first treat the case where $A$ is a $Q$-ring. Then $A$

contains a compact open radical subring $B$. Let $M$ be an open maximal ideal in $A$, so that $A/M$ is a discrete division ring. Then $B$ maps into a finite radical subring of $A/M$; such a subring is a nilring and hence 0. Thus $B$ is contained in $M$. If we denote by $I$ the intersection of all open maximal ideals, then $I$ is open since it contains $B$. We proceed to examine $I$, which, as a ring on its own merits is again strongly regular. Its maximal ideals are in one-to-one correspondence with the non-open maximal ideals of $A$. Thus for every maximal ideal $N$ in $I$, $I/N$ is a non-discrete locally compact division ring. We shall now prove that there are only a finite number of $N$'s. If we suppose the contrary, then $I$ has an infinite set of orthogonal idempotents $e_i$ (this follows from the results in [1], or it can be proved by a simple direct argument). For each $i$ we can find an element $b_i$ in $B$ such that $e_ib_i$ is a non-zero element of $B$. Write $c_n = e_1b_1 + \cdots + e_nb_n$, $a_n = e_1b_1 + e_2b_2^2 + \cdots + e_nb_n^n$ and let $c,a \in B$ be limit points of the sequences $c_n$, $a_n$ respectively. Find the element $x$ with $a^2x = a$. Now $c^r \to 0$ since $B$ is topologically nilpotent; hence $xc^r \to 0$, $xc^r \in B$ for a suitably large $r$. Take any $s > r$, and examine the situation modulo a maximal ideal $N$ in $I$ not containing $e_sb_s$. The division ring $I/N$ has a unique valuation associated with it which we shall denote by $V$, and we may apply $V$ in a natural way to all elements of $I$. Since $e_sb_s$ is topologically nilpotent, $V(b_s) > 0$. Next we note $e_sc_n = e_sb_s$ for all $n \geqq s$ and hence $e_sc = e_sb_s$; similarly $e_sa = e_sb_s^s$. From $a^2x = a$ we reduce $V(x) = -sV(b_s)$. On the other hand, from the fact that $xc^r$ is in $B$ we deduce $V(x) + rV(b_s) > 0$, a contradiction.

At this point we have proved that $I$ is the direct sum of a finite number of non-discrete division rings. Thus it has a unit element and is a direct summand of $A$; moreover $A$ is the direct sum of $I$ and the discrete ring $A/I$. This concludes the discussion of the case where $A$ is a $Q$-ring.

In the general case we apply Theorem 2 (applicable since the idempotents are in the center) to learn that $A$ is a local direct sum of $Q$-rings. Each summand is strongly regular and its structure is known from the preceding discussion. From this we can see that all but a finite number of summands must be discrete—otherwise the local sum would not be strongly regular. In the local sum over the discrete summands, all but a finite number of the subrings (relative to which the local sum is taken) must be finite. Those summands for which the subring in question is 0 may be lumped together with the finite number of exceptions already encountered to form a discrete summand. The remaining local sum is taken relative to finite non-zero subrings all but a finite number of which must be strongly

regular. Since a finite strongly regular ring is a direct sum of finite fields, the sum may be further decomposed so as to be taken to finite fields (the finite number of exceptions are again moved over to the discrete summand).

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# PRIME IDEALS AND THE LOWER RADICAL.*

By JAKOB LEVITZKI.

The lower radical $L$ of a ring $R$ has been defined by Baer [1][1] as the intersection of all the radical ideals[2] of the ring. Baer has shown that $L$ itself is a radical ideal. The lower radical of an ideal[3] $A$ may be defined as the ideal $L(A)$ which is uniquely determined by the following two properties: 1) $A \subseteq L(A)$. 2) $L(A)/A$ is the lower radical of $R/A$. Denote by $M(A)$ the radical of $A$ in the sense of McCoy [4], i. e. the set of all elements $r$ with the property that every $m$-system[4] which contains $r$ contains an element of $A$. McCoy has shown that $M(A)$ is an ideal, $M(A)$ is the intersection of all the prime ideals which contain $A$ and finally that $M(A)/A$ is a radical ideal in $R/A$. The radical M of the ring is defined as the radical of the zero ideal. McCoy remarks that his radical $M(A)$ "is contained in Fitting's radical, but the exact relation between these concepts is an unsolved problem." He also says that "the relation of $M$ to the radicals of Koethe and Levitzki is an unsolved problem." In the present note we solve these problems by showing that *the radical ideal $M(A)/A$ of $R/A$ is actually the lower radical of $R/A$, i. e.* $M(A) = L(A)$. In particular $M = L$. This implies that $M$ is semi-nilpotent and that $M$ is distinct from the maximal semi-nilpotent ideal of the ring, and hence also from the radical in the sense of Koethe (see [1], [2], and [3]).[5] The same applies to the radical of an ideal. On the other hand McCoy's results combined with our theorem yield the following characterization of the lower radical: *The lower radical of a ring is the intersection of all the prime ideals of the ring.* In the following we use the concept of an $m$-sequence, which is actually a special case of an $m$-

* Received January 11, 1950.

[1] Numbers in brackets [ ] refer to the bibliography at the end of the paper.

[2] An ideal $A$ is a radical ideal if $A$ is a nil-ideal and $R/A$ has no nil-potent ideals other than zero.

[3] Ideal = two sided ideal.

[4] A set $S$ is an $m$-system if $a \in S$, $b \in S$ implies the existence of $c \in R$ such that $acb \in S$.

[5] In an example given by Baer ([1], § 2) we have $R \neq 0$, $L = 0$ while the radical in the sense of Koethe etc. is $= R$.

system as defined by McCoy. Incidentally this concept yields yet another characterization of the lower radical.[6]

We begin with the following definitions:

*Definition* 1. An infinite sequence of elements in a ring $R$

$$a_0, a_1, a_2, \cdots \tag{1}$$

is an $m$-sequence if for each non-negative integer $n$ there exists an element $b_n$ in $R$ so that $a_{n+1} = a_n b_n a_n$.

*Definition* 2. An $m$-sequence is a vanishing $m$-sequence if some term of the sequence is $= 0$.

*Remark.* If some term of an $m$-sequence is $= 0$, then evidently also all subsequent terms are $= 0$. Hence a vanishing $m$-sequence contains at most a finite number of nonzero terms. Moreover, since each term of an $m$-sequence is a left factor as well as a right factor of all subsequent terms, we have the following

LEMMA 1. *If some term of an m-sequence is contained in a right (left) ideal A, then also all subsequent terms are contained in A.*

We need also the following

LEMMA 2. *The elements of an m-sequence form an m-system.*

*Proof.* Consider an $m$-sequence (1). We have to show that for any pair of terms $a_i$, $a_k$ of (1) there exists a term $a_j$ and an element $b_{ik}$ of $R$ such that $a_j = a_i b_{ik} a_k$. Now it is an easy consequence of the defining relation $a_{n+1} = a_n b_n a_n$ that this holds for any $a_j$ with $j > i$, $j > k$.

*Definition* 3. An element $a$ of a ring $R$ and an infinite sequence $b_0, b_1, \cdots$ are associated with each other if $b_0 = a$ and for some infinite sequence $c_0, c_1, \cdots$ we have $b_{n+1} = b_n c_n b_n$, $i = 0, 1, 2, \cdots$.

As an immediate consequence of definitions 1 and 3 we obtain

LEMMA 3. *Each infinite sequence which is associated with some element of a ring R is an m-sequence.*

*Remark.* Conversely, each $m$-sequence (1) is evidently associated with $a_0$. Hence Lema 3 gives a method for constructing all possible $m$-sequences.

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[6] As will be shown elsewhere a certain generalization of this concept may be used with advantage also in other problems concerning nil-rings.

THEOREM 1. *An element $a$ of a ring $R$ belongs to the lower radical $L$ of $R$ if and only if each sequence which is associated with $a$ is a vanishing $m$-sequence.*

*Proof.* For the proof of the "only if" part of the theorem we need the following characterization of the lower radical: Denote by $N_1$ the ordinary radical of $R$, i. e. the sum of all nilpotent ideals of $R$. If $\alpha > 1$ is an ordinal such that $\alpha = \beta + 1$, then the ideal $N_\alpha$ is defined by the relation $N_\beta \subseteq N_\alpha$ and the requirement that $N_\alpha/N_\beta$ should be the ordinary radical of $R/N_\beta$. If $\alpha$ is a limit ordinal, then $N_\alpha$ is defined as the join of all $N_\gamma$ with $\gamma < \alpha$. For some ordinal $\tau$ of minimal value we have then $N_{\tau+1} = N_\tau$ and it follows that $L = N_\tau$ (see [1], page 539). Suppose now that $a \in L$ and consider a sequence (1) which is associated with $a$, so that we have $a_0 = a$. By Lemma 1 it follows that all terms of (1) are elements of $L$. Hence each element of (1) is contained in some $N_\gamma$. Thus for each non-negative integer $i$ there exists a smallest ordinal $\alpha_i$ so that $a_i \in N\alpha_i$. If for some $i$ the ordinal $\alpha_i$ were a limit ordinal, it would follow from the definition of $N\alpha_i$ as the join of all $N_\gamma$ with $\gamma < \alpha_i$, that $a_i \in N_\gamma$ for some $\gamma < \alpha_i$, which is a contradiction to the minimum property of $\alpha_i$. Hence none of the ordinals $\alpha_i$ is a limit ordinal. Denote by $k$ a non-negative integer so that $\alpha_k = \text{minimum } \alpha_i$, $i = 0, 1, 2, \cdots$. Since $\alpha_k$ is not a limit ordinal we have either $\alpha_k = 1$ or $\alpha_k = \beta + 1$ with $\beta \geq 1$. Suppose first that we are confronted by the second possibility, then by the definition of $N_{\beta+1}$ it follows that for some positive integer $m$ the right ideal $(a_k, a_kR)$ generated by $a_k$ satisfies the relation:

$$(a_k, a_kR)^m \subseteq N_\beta. \tag{2}$$

Now we have $a_{k+1} = a_kb_ka_k \in (a_k, a_kR)^2$, and for an arbitrary positive integer $p$ it follows easily that $a_{k+p} \in (a_k, a_kR)^{2^p}$. Hence by choosing $p$ so that $2^p \geq m$ we obtain

$$a_{k+p} \in (a_k, a_kR)^{2^p} \subseteq (a_k, a_kR)^m \subseteq N_\beta. \tag{3}$$

Relation (3) shows that $\alpha_{k+p} \leq \beta < \alpha_k$, which is a contradiction to the minimum property of $\alpha_k$. This contradiction implies that we must have $\alpha_k = 1$ and thus relations (2) and (3) may be replaced by $(a_k, a_kR)^m = 0$ and $a_{k+p} = 0$ respectively, which shows that each sequence which is associated with $a$ is a vanishing $m$-sequence. This concludes the proof of the "only if" part of the theorem. The "if" part will be settled if we can show that any element $a$ such that $a \notin L$ is associated with a non-vanishing $m$-sequence. We begin the construction of such a sequence by setting $a_0 = a$. Since $L$ is a

radical ideal, i. e., $R/L$ has no nilpotent ideals, it follows by $a_0 \notin L$ that also $a_0R \cdot a_0R \notin L$. Hence for some $b_0 \in R$ we have $a_0b_0a_0 \notin L$. We define the second term of the required sequence by putting $a_1 = a_0b_0a_0$. Since $a_1 \notin L$ we may repeat the above procedure with $a_1$ and thus obtain the third term $a_2 = a_1b_1a_1 \notin L$. By continuing this process we obtain (by induction) a non-vanishing $m$-sequence which is associated with $a$. This completes the proof of the theorem.

THEOREM 2. *The radical of a ring $R$ in the sense of McCoy coincides with the lower radical $L$.*

*Proof.* Denote by $M$ the radical in the sense of McCoy and suppose first that for some element $a$ we have $a \in M$, $a \notin L$. By the "if" part of Theorem 1 it follows that there exists a non-vanishing $m$-sequence which is associated with $a$. By Lemmas 2 and 3 we known that the elements of such a sequence form an $m$-system. Since this system contains $a$ and does not contain 0 we have a contradiction to $a \in M$. This shows that if $a \in M$ then also $a \in L$, i. e. $M \subseteq L$. Suppose now [7] that there exists an element $a$ such that $a \in L$, $a \notin M$. By the definition of $M$ it follows that there exists an $m$-system $S$ which contains $a$ and does not contain 0. Hence for some $b_0 \in R$ we have $0 \neq ab_0a \in S$. Put $a_0 = a$ and $a_1 = a_0b_0a_0$. Since $a_1 \in S$ we may repeat the above procedure with $a_1$, and thus by continuing the process obtain (by induction) a non-vanishing $m$-sequence which is associated with $a$. Since by Theorem 1 the existence of such a sequence is incompatible with $a \in L$ it follows that we must have $a \in M$, or $L \subseteq M$ which in conjunction with $M \subseteq L$ yields the required result.

COROLLARY 1. *For an ideal $A$ of a ring $R$ we have $M(A) = L(A)$.*

By combining Theorem 2 with McCoy's results [4] we obtain

COROLLARY 2. *The lower radical of a ring is the intersection of all the prime ideals of the ring.*

Since the lower radical of a ring is $= 0$ if and only if the zero ideal is the only nilpotent ideal of the ring we have by [4], Theorem 6:

COROLLARY 3. *A necessary and sufficient condition that a ring be iso-*

[7] The proof may be completed as follows: Since $M$ is a radical ideal (see [4]) we have $M \supseteq L$, which in conjunction with $L \supseteq M$ yields $M = L$. Now the relation $M \supseteq L$ has been proved by McCoy by using the concept and the properties of a prime ideal. The proof of $M \supseteq L$ in the text does not involve prime ideals and uses the "only if" part of Theorem 1 instead.

*morphic to a subdirect sum of prime rings is that the given ring does not contain nilpotent ideals other than zero.*

Finally we have

COROLLARY 4. *The intersection of all the prime ideals of a ring is* $= 0$ *if and only if the zero ideal is the only nilpotent ideal of the ring.*

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# BANACH ALGEBRAS OF CONTINUOUS FUNCTIONS.*

By Bertram Yood.

**1. Introduction.** For the Banach algebra $C(X)$ of all bounded, continuous and complex-valued functions defined on a completely regular space with the usual definition of the norm, the space of maximal ideals is homeomorphic to the Čech $\beta$-compactification of $X$ [6]. In this paper we investigate the structure of the space of maximal ideals of the more general Banach algebras $C(X, A)$ of all bounded continuous functions defined on a space $X$ as above with values in a commutative complex Banach algebra $A$ with a unit.

In § **3** it is shown that if $A$ is also a $B^*$-algebra (in the sense of [11]) then the space of maximal ideals of $C(X, A)$ is homeomorphic to the $\beta$-compactification of the Cartesian product space $X \times \mathfrak{M}$, where $\mathfrak{M}$ is the space of maximal ideals of $A$. This situation is further investigated in § **4** under the additional assumptions that $X$ and $X \times \mathfrak{M}$ are normal. This yields other descriptions of the space of maximal ideals of $C(X, A)$ and hence alternative descriptions of the $\beta$-compactification of $X \times \mathfrak{M}$. This is given in terms of the $\beta$-compactification of $X$, the Cartesian product space $P_{t \in X} \mathfrak{M}$ and continuous functions on $X \times \mathfrak{M}$ and closure in $X \times \mathfrak{M}$.

In § **5** it is assumed that $X$ is compact (= bicompact) but the requirement that $A$ be a $B^*$-algebra is dropped. Here it is shown that $C(X, A)$ may have $X$ as its space of maximal ideals without $A$ being the algebra of scalars, specifically for $A$ a primary ring (i. e., a ring with one maximal ideal).

An investigation of rings $C(X, A)$ of all continuous functions on a topological space $X$ to a topological ring $A$ was made by Kaplansky [10]. Where the hypotheses are comparable, the results of Kaplansky hold for much more general rings $A$ but much greater demands are made on the spaces $X$ than in this paper. Kaplansky also considers rings of all continuous functions (unbounded as well as bounded). Here we use only bounded functions and endeavor to exploit as fully as possible the techniques of commutative Banach algebra theory as developed by Gelfand and others.

A thorough exposition of the theory of the algebra $C(X)$ (for the real case where the theory is quite the same) can be found in a paper of Hewitt

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[7]. Hewitt also investigates the algebra of all (bounded and unbounded) continuous real-valued functions defined on the completely regular space $X$.

**2. Definitions and preliminaries.** We use the letter $A$ to represent a complex Banach algebra (normed ring) which is commutative and posseses a unit $e$ of norm one. For the fundamental properties of $A$ see [3] or [9]. Associated with $A$ is its space $\mathfrak{M}$ of maximal ideals topologized as in [3] to become a compact Hausdorff sapce. Each $M \in \mathfrak{M}$ determines a homomorphism $X(M)$ of $A$ onto the complex numbers $K$. $M$ consists of all $x \in A$ for which $x(M) = 0$ and for each $x$, $x(M)$ is a continuous function of $M$. As a function of $x$ for a fixed $M$, $x(M)$ is a multiplicative linear functional defined on $A$ and all such functionals are of this form.

For a topological space $X$, let $C(X, A)$ be the collection of all bounded continuous functions defined on $X$ with values in $A$. With the norm $\| f(\cdot) \| = \sup \| f(t) \|$, $t \in X$, $C(X, A)$ is a commutative Banach algebra with a unit.

2.1. LEMMA. *Let $t_0$ be a point of $X$ and $M$ a maximal ideal of $A$. Then the set $N$ of all $f(\cdot) \in C(X, A)$ for which $f(t_0) \in M$ forms a maximal ideal of $C(X, A)$.*

Suppose that $g(\cdot) \in C(X, A)$ and $g(t_0) \notin M$. Then there exist elements $v \in A$, $w \in M$ such that $vg(t_0) + w = e$. We let $h(\cdot) \in C(X, A)$ be the function with the constant value $v$ and $e(\cdot)$ the function with the constant value $e$. Then

$$e(\cdot) = h(\cdot)g(\cdot) + \{e(\cdot) - h(\cdot)g(\cdot)\}$$

where $e(\cdot) - h(\cdot)g(\cdot) \in N$ since $e - vg(t_0) = w$. This shows that $g(\cdot)$ and $N$ generate the ideal $C(X, A)$ and therefore that $N$ is maximal.

On the basis of this result we divide the ideals into two classes which are the analogues of those used by Hewitt [7]. An ideal $I$ in $C(X, A)$ is called *fixed* if there exist $t_0 \in X$ and $M \in \mathfrak{M}$ such that

$$I \subset \{f(\cdot) \in C(X, A) \mid f(t_0) \in M\}.$$

Otherwise the ideal $I$ is called *free*.

For $f(\cdot) \in C(X, A)$, $t \in X$, $M \in \mathfrak{M}$, we write $f(t)(M)$ as $f(t, M)$.

2.2. LEMMA. *For each $f(\cdot) \in C(X, A)$, $f(t, M)$ is a continuous function on the Cartesian product space $X \times \mathfrak{M}$.*

Let $(t_0, M_0)$ be a point of $X \times \mathfrak{M}$ and $\epsilon > 0$. Now

$$| f(t, M) - f(t_0, M_0) | \leqq | f(t, M) - f(t_0, M) | + f(t_0, M) - f(t_0, M_0) |$$

and $|f(t,M)-f(t_0,M)| \leqq \|f(t)-f(t_0)\|$. We choose a neighborhood $U(t_0)$ of $t_0$ in $X$ for which $\|f(t)-f(t_0)\| < \epsilon/2$, and a neighborhood $V(M_0)$ of $M_0$ in $\mathfrak{M}$ for which $|f(t_0,M)-f(t_0,M_0)| < \epsilon/2$. For $(t,M)$ in $U(t_0) \times V(M_0)$ we have $|f(t,M)-f(t_0,M_0)| < \epsilon$.

Following Rickart [11] we call a Banach algebra a "$B^*$-algebra" if to each element $x$ there corresponds a unique element $x'$, called the adjoint of $x$ where (i) $(x')' = x$, (ii) $(xy)' = y'x'$, (iii) for $\alpha$ and $\beta$ complex numbers and $\bar{\alpha}$ and $\bar{\beta}$ their complex conjugates, $(\alpha x + \beta y)' = \bar{\alpha}x' + \bar{\beta}y'$, (iv) $\|x'x\| = \|x\|^2$.

The $B^*$-algebras considered in this paper are commutative and possess a unit of norm one. It is known that $\|x'\| = \|x\|$ and that such a $B^*$-algebra $A$ is equivalent to the algebra of all complex-valued continuous functions defined on its space of maximal ideals $\mathfrak{M}$ ([5, Lemma 1]). Also $\|x\| = \sup |x(M)|$, $M \in \mathfrak{M}$, $x'(M) \equiv \overline{x(M)}$ for each $M \in \mathfrak{M}$, and $A$ is without radical. As the topology for $\mathfrak{M}$, we use that of [3] in which the neighborhoods of $M_0 \in \mathfrak{M}$ are all sets of the form

$$\{M \in \mathfrak{M} \mid |x_i(M) - x_i(M_0)| < \epsilon,\ i = 1, \cdots, n\}$$

where $x_1, \cdots, x_n$ are elements of $A$ and $\epsilon > 0$. This yields a compact Hausdorff topology for $\mathfrak{M}$. Two other topologies are investigated by Gelfand and Šilov in [6]. The first one, introduced by Stone [15], may be described as follows. For each set $\mathfrak{A} \subset \mathfrak{M}$, $M_0 \in \mathfrak{A}$ if and only if $M_0 \supset \bigcap M$, $M \in \mathfrak{A}$. The other topology can be employed in any commutative Banach algebra $A$ with a unit for which given $x \in A$ there is $y \in A$ with $x(M) = \overline{y(M)}$, for every $M \in \mathfrak{M}$. Here $M_0 \in \mathfrak{A}$ if and only if $M_0 \subset \overline{\bigcup M}$, the union taken over $M \in \mathfrak{A}$. Equivalently $M_0 \in .\mathfrak{A}$ if and only if for every $x \in M_0$, $\inf |x(M)| = 0$, $M \in \mathfrak{A}$. This topology can be employed for our $B^*$-algebras. In view of results in [6], the theorem of Gelfand [3] which states that any compact Hausdorff topology for $\mathfrak{M}$ in which all the functions $x(M)$ are continuous is homeomorphic to $\mathfrak{M}$ in the first topology above and the fact that a $B^*$-algebra is equivalent to $C(\mathfrak{M})$ ($\mathfrak{M}$ with the first topology), all three topologies for $\mathfrak{M}$ yield homeomorphic spaces.

For such a $B^*$-algebra $A$, $C(X,A)$ is also a $B^*$-algebra to which the above remarks about its maximal ideals apply.

**3. $X$ completely regular, $A$ a $B^*$-algebra.** We obtain the structure of the space of maximal ideals of $C(X,A)$ in this case as the $\beta$-compactification in the sense of Čech [2] of the completely regular space $X \times \mathfrak{M}$. The compactification procedure of Čech is equivalent to the compactification procedure of Gelfand and Šilov [6].

3.1. THEOREM. *Let $X$ be a completely regular space, $A$ a commutative $B^*$-algebra with a unit. Then the space of maximal ideals of $C(X, A)$ is homeomorphic to the $\beta$-compactification of $X \times \mathfrak{M}$.*

In Lemma 2.2 it was shown that for any $f(\cdot) \in C(X, A)$, $f(t, M)$ is continuous on $X \times \mathfrak{M}$. Conversely we show that if $f(t, M)$ is continuous on $X \times \mathfrak{M}$, there exists $g(\cdot) \in C(X, A)$ such that $f(t, M) = g(t, M)$ for each $t \in X$, $M \in \mathfrak{M}$. Let $t_0 \in X$. For each $M' \in \mathfrak{M}$, there exists a neighborhood $V(M')$ of $M'$ and a neighborhood $U(M', t_0)$ of $t_0$ in $X$ such that for all $t \in U(M', t_0)$, $M \in V(M')$ we have $|f(t, M) - f(t_0, M')| < \epsilon$. A finite number $V(M_1), \cdots, V(M_n)$ of the sets $V(M')$ cover $\mathfrak{M}$. Let $U(t_0) = \bigcap U(M_i, t_0)$. Then for each $M \in \mathfrak{M}$, $|f(t, M) - f(t_0, M)| < \epsilon$ for $t \in U(t_0)$.

Next the function $g(\cdot)$ is defined for $t = t_0$. $g(t_0)$ is taken to be that element in $A$ such that for each $M \in \mathfrak{M}$, $g(t_0)(M) = f(t_0, M)$. Since $f(t_0, M)$ is continuous in $M$ and $A$ is a $B^*$-algebra, this is possible. This procedure is followed for each element in $X$. Also $g(\cdot) \in C(X, A)$. For given $\epsilon > 0$ there is a neighborhood $U(t_0)$ such that for $t \in U(t_0)$,

$$\| g(t) - g(t_0) \| = \sup_{M \in \mathfrak{M}} | g(t)(M) - g(t_0)(M) |$$
$$= \sup | f(t, M) - f(t_0, M) | \leqq \epsilon$$

by the above analysis. ($f(t, M)$ is assumed bounded.)

The correspondence $f(\cdot) \to f(t, M)$ between $C(X, A)$ and $C(X \times \mathfrak{M})$, the Banach algebra of all complex-valued, bounded, continuous functions defined on $X \times \mathfrak{M}$, is next considered. It is easy to verify that this is a 1-1 correspondence of $C(X, A)$ onto $C(X \times \mathfrak{M})$ which preserves all the algebraic operations. Furthermore since

$$\| f(\cdot) \| = \sup_{t \in X} \| f(t) \| = \sup_{t \in X} \sup_{M \in \mathfrak{M}} | f(t, M) |$$

and since

$$\sup_{M \in \mathfrak{M}} | f(t, M) | = | f(t, M_t) |$$

for a suitably chosen $M_t \in \mathfrak{M}$,

$$\| f(\cdot) \| = \sup_{t \in X} | f(t, M_t) | = \sup_{(t, M) \in X \times \mathfrak{M}} | f(t, M) | ,$$

the correspondence is an equivalence. Now $X \times \mathfrak{M}$ is completely regular and hence the space of maximal ideals of $C(X \times \mathfrak{M})$ is homeomorphic to the $\beta$-compactification of $X \times \mathfrak{M}$ (see [6]). This is then the case for the space of maximal ideals of $C(X, A)$.

**4. $X$ and $X \times \mathfrak{M}$ normal, $A$ a $B^*$-algebra.** Under these conditions we give other descriptions of the space of maximal ideals of $C(X, A)$. By

Theorem 3.1, this then gives us descriptions of the $\beta$-compactification of $X \times \mathfrak{M}$. In this we use the points of the Wallman compactification of $X$ (which is equivalent to the $\beta$-compactification since $X$ is normal), the Cartesian product space $P_{t \in X} \mathfrak{M}$ and the continuous functions on $X \times \mathfrak{M}$. The points of the compactification are pairs of elements of the compactification of $X$ and equivalence in the Cartesian product space.

We wish to emphasize the fact that the more complicated description of the space of maximal ideals of $C(X, A)$ which is given in this section has the virtue (not possessed by the analysis of § 3) that it yields the general form of a multiplicative linear functional defined on $C(X, A)$.

For $f(\cdot) \in C(X, A)$ we define the set

$$(4.1) \qquad X(f, \epsilon) = \{t \in X \mid |f(t, M)| \leqq \epsilon \text{ for at least one } M \in \mathfrak{M}\}.$$

Let $I$ be a non-trivial ideal of $C(X, A)$. We denote by $\mathfrak{A}(I)$ the collection of all sets of the form $X(f, \epsilon)$, $f \in I$, for a fixed $\epsilon > 0$. Inasmuch as $I$ is a linear manifold, $\mathfrak{A}(I)$ is independent of the choice of $\epsilon > 0$ in its definition.

4.1. Lemma. *The family of sets $\mathfrak{A}(I)$ has the finite intersection property.*

Let $B(X, A)$ be the $B^*$-algebra of all bounded functions defined on $X$ with range in $A$ where $\| f(\cdot) \| = \sup \| f(t) \|$, $t \in X$. Then $C(X, A)$ is a $B^*$-subalgebra of $B(X, A)$. $I$ is contained in a maximal ideal $N$ of $C(X, A)$. By a theorem of Šilov [12, p. 84], $N$ is contained in a maximal ideal $N_1$ of $B(X, A)$. If we define the sets $X(f, \epsilon)$ for each $f \in N_1$ then the collection so obtained has the finite intersection property [18, Lemma 2.2]. Since $\mathfrak{A}(I)$ is a subcollection, it shares this property.

4.2. Lemma. *$\mathfrak{A}(I)$ is contained in a maximal additive ideal in the distributive lattice of all closed subsets of $X$.*

We show first that the sets $X(f, \epsilon)$ are closed. Let $t_0$ be in the closure of $X(f, \epsilon)$. Suppose that $\inf |f(t_0, M)| = \eta > \epsilon$, $M \in \mathfrak{M}$. Then there is a neighborhood $U(t_0)$ of $t_0$ such that for each $t \in U(t_0)$, $\| f(t) - f(t_0) \| < (\eta - \epsilon)$. For each such $t$ and each $M \in \mathfrak{M}$ we have

$$|f(t, M)| \geqq |f(t_0, M)| - |f(t_0, M) - f(t, M)| > \epsilon$$

inasmuch as $|f(t_0, M) - f(t, M)| \leqq \| f(t_0) - f(t) \|$. This is a contradiction. Since $f(t_0, M)$ is a continuous function on the compact set $\mathfrak{M}$, there is an $M \in \mathfrak{M}$ for which $|f(t_0, M)| \leqq \epsilon$ and $X(f, \epsilon)$ is closed.

Thus $\mathfrak{A}(I)$ having the finite intersection property is contained in a maximal additive ideal in the distributive lattice of closed subsets of $X$; see [17, p. 114]. We use the letter $\mathfrak{L}$ to designate this lattice. By an

*additive ideal* $\mathfrak{B}$ *in* $\mathfrak{L}$ is meant a collection which has the properties that if $F \in \mathfrak{B}$, $G \in \mathfrak{L}$, then $F \cup G \in \mathfrak{B}$ and if $F \in \mathfrak{B}$, $G \in \mathfrak{B}$ then $F \cap G \in \mathfrak{B}$. A *maximal additive ideal* (m. a. ideal) is one which is contained in no proper additive ideal of $\mathfrak{L}$.

We designate by $\mathfrak{S}$, the Boolean algebra of subsets of $X$ which is generated by the open sets and the closed set of $X$. See [1] for the lattice-theoretic terminology employed. The following lemma on distributive lattices is needed.

4. 3. LEMMA. *Let* $\mathfrak{A}$ *be a lattice with a zero element* $\Lambda$. *Let* $\mathfrak{B}$ *be a maximal additive ideal in* $\mathfrak{A}$. *A real valued function* $\mu$ *defined on* $\mathfrak{A}$ *for which* $\mu(E) = 1$, $E \in \mathfrak{B}$, $\mu(E) = 0$, $E \notin \mathfrak{B}$ *is additive in the sense that if* $E_1 \cap E_2 = \Lambda$, *then* $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$.

It is to be assumed that $\mathfrak{A}$ is not the trivial lattice consisting of just its zero element. Wallman [7] has shown that for $\mathfrak{B}$ to be a m. a. ideal it is necessary and sufficient that $\mathfrak{B}$ be a maximal collection of elements with the finite intersection property, and that this is equivalent to the possession by $\mathfrak{B}$ of the properties (1) if $F, G \in \mathfrak{B}$ then $F \cap G \in \mathfrak{B}$ and (2) $F \in \mathfrak{B}$ if and only if $F \cap G \neq \Lambda$ for every $G \in \mathfrak{B}$. From this it follows readily if $E_1 \in \mathfrak{B}$ and $E_1 \cap E_2 = \Lambda$ then $E_2 \in \mathfrak{B}$, $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$. Also if $\mu(E_1) = \mu(E_2) = 0$ where $E_1 \cap E_2 = \Lambda$, there exist elements $F_i \in \mathfrak{B}$ such that $E_i \cap F_i = \Lambda$, $i = 1, 2$. But then $(F_1 \cap F_2) \cap (E_1 \cup E_2) = \Lambda$. However $F_1 \cap F_2 \in \mathfrak{B}$ so that $E_1 \cup E_2 \notin \mathfrak{B}$.

The converse is true if the lattice is a Boolean algebra.

4. 4. LEMMA. *Let* $\mathfrak{A}$ *be a Boolean algebra on which there is defined a non-trivial real-valued additive function* $\mu$ *taking on the values zero and one only. Then the set* $\mathfrak{B}$ *of elements of* $\mathfrak{A}$ *for which* $\mu(E) = 1$ *forms a maximal additive ideal in* $\mathfrak{A}$.

Let $E_1, E_2 \in \mathfrak{B}$. Using the prime to denote complementation we have $E_1 = (E_1 \cap E_2) \cup (E_1 \cap E'_2)$ and $\mu(E_1) = \mu(E_1 \cap E_2) + \mu(E_1 \cap E'_2)$. Thus one and only one of the sets $E_1 \cap E_2$ and $E_1 \cap E'_2$ are in $\mathfrak{B}$. Clearly $\mu(E'_2) = 0$ so that since $E'_2 = (E_1 \cap E'_2) \cup (E'_1 \cap E'_2)$ each of the latter sets has $\mu$-measure zero. Hence $E_1 \cap E_2 \in \mathfrak{B}$. Clearly $\Lambda \notin \mathfrak{B}$. If $F \in \mathfrak{B}$, $G \in \mathfrak{A}$ then it is easy to see that $F \cup G \in \mathfrak{B}$. Hence $\mathfrak{B}$ is an additive ideal in $\mathfrak{A}$, hence, by Zorn's maximum principle, contained in an m. a. ideal $\mathfrak{B}_1$ of $\mathfrak{A}$. Let $\mu_1$ be defined on $\mathfrak{A}$ to be 1 for elements on $\mathfrak{B}_1$ and zero otherwise. By Lemma 4. 3, $\mu_1$ is additive. Also $\mu_1$ agrees with $\mu$ on elements in $\mathfrak{B}$ and thus on the complements of such elements. Since any element is either in $\mathfrak{B}$ or is a complement of such an element, $\mu_1 \equiv \mu$ and $\mathfrak{B} = \mathfrak{B}_1$.

4.5. LEMMA. *Each maximal additive ideal $\mathfrak{B}$ of $\mathfrak{L}$ is contained in one and only one maximal addtive ideal $\mathfrak{B}_1$ of $\mathfrak{S}$.*

$\mathfrak{B}$ is a maximal family of closed subsets of $X$ having the finite intersection property, as mentioned above. By Zorn's maximum principle $\mathfrak{B}$ is contained in some maximal collection $\mathfrak{B}_1$ of sets in $\mathfrak{S}$ with the finite intersection property. By Wallman [17], $\mathfrak{B}_1$ is a m. a. ideal of $\mathfrak{S}$. Suppose $\mathfrak{B}_2$ is another m. a. ideal of $\mathfrak{S}$ containing $\mathfrak{B}$. By Lemma 4. 3, the set functions $\nu_i$ defined on $\mathfrak{S}$, one for sets in $\mathfrak{B}_i$, zero otherwise are additive, $i = 1, 2$. Moreover for each $i$, with $E_1, E_2 \in \mathfrak{S}$, if $\nu_i(E_1) = \nu_i(E_2) = 1$, then

$\nu_i(E_1 \cup E_2) = \nu_i(E_1 \cap E_2) = 1$ since $\mathfrak{B}_i$ is an additive ideal in $\mathfrak{S}$.
For any $E$, $\nu_i(E') = 1 - \nu_i(E)$. If $\nu_i(E_1) = 1$, $\nu_i(E_2) = 0$, then $\nu_i(E_1 \cup E_2) = 1$. Also since $1 = \nu_i(E_1) = \nu_i(E_1 \cap E_2) + \nu_i(E_1 \cap E'_2)$ and $\nu_i(E_1 \cap E'_2) = 1$ by the preceding, we have $\nu_i(E_1 \cap E_2) = 0$. If $\nu_i(E_1) = \nu_i(E_2) = 0$, then $\nu_i(E_1 \cap E_2 = 1 - \nu_i(E'_1 \cup E'_2) = 0$, $\nu_i(E_1 \cup E_2) = 1 - \nu_i(E'_1 \cap E'_2) = 0$. Also for $F \in \mathfrak{B}$, $\nu_i(F) = 1$. For any closed set $G \notin \mathfrak{B}$, if $\nu_i(G) = 1$, then $G \cap F \neq \Lambda$ for every $F \in \mathfrak{B}$ and thus the family $\mathfrak{B}$ may be augmented by $G$ preserving the finite intersection property, which is impossible. Since for any open set, $O$, $\nu_i(O) = 1 - \nu_i(O')$, $\nu_1 = \nu_2$ for any open or closed set in $X$. Since $\mathfrak{S}$ is the Boolean algebra of subsets of $X$ generated by these sets, the above rules for $\nu_i$ show that $\nu_1 = \nu_2$ for any set in $\mathfrak{S}$. Thus $\mathfrak{B}_1 = \mathfrak{B}_2$.

4. 6. LEMMA. *Let $N$ be a maximal ideal of $C(X, A)$ and $f_i \in N$, $i = 1, 2$. Then there exists a function $h \in N$ with the property that for each $t \in X$, $M \in \mathfrak{M}$,*

$$h(t, M) = \sup_{i=1,2} | f_i(t, M) | .$$

To prove this one may proceed as in Lemma 2. 4 of [18]. With the notation used there, it is only necessary to note the function $k(\cdot)$ appearing there is continuous. This follows since

$$\| k(t_1) - k(t_2) \| = \sup_{M \in \mathfrak{M}} | \, | f(t_1, M) | - | f(t_2, M) | \, | \leqq \| f(t_1) - f(t_2) \| .$$

Clearly this result holds for any finite number of functions in $N$.

In the following lemma and its proof, the following notation is employed. $N$ is a maximal ideal of $C(X, A)$ and $\mathfrak{A}(N)$ is the corresponding family of closed sets in $X$ previously defined. $\Gamma$ is the collection of all single-valued functions defined on $X$ where for each $t \in X$, the image of $t$ is in $\mathfrak{M}$. For $\gamma \in \Gamma$, $f(\cdot) \in C(X, A)$, $\mathfrak{A}(f, \gamma, \epsilon)$ is the set $\{t \in X \mid | f(t, \gamma(t)) | \leq \epsilon\}$. For $f(\cdot) \in N$, $\Gamma(f, \epsilon) = \{\gamma \in \Gamma \mid \mathfrak{A}(f, \gamma, \epsilon)$ contains a set in $\mathfrak{A}(N)\}$. $\Delta$ is the collection of all sets $\Gamma(f, \epsilon)$ as $f(\cdot)$ ranges over $N$. $\mathfrak{B}$ is a m. a. ideal in $\mathfrak{L}$ containing $\mathfrak{A}(N)$, $\mathfrak{B}_1$ is the (unique) m. a. ideal of $\mathfrak{S}$ containing $\mathfrak{B}$. $\mu$ is a

function defined on $\mathfrak{S}$ to be one for elements in $\mathfrak{B}_1$, zero otherwise. By Lemmas 4. 3 and 4. 4, $\mu$ is additive and there is the sole additive set function which is one for sets in $\mathfrak{B}$ and zero for all other set in $\mathfrak{L}$. We say that $\mu$ is *the* $0-1$ *measure function associated with the m. a. ideal* $\mathfrak{B}$ of $\mathfrak{L}$.

The integral

$$\int_X f(t, \gamma(t))\, d\mu \tag{4.2}$$

taken in the sense of Hildebrandt [8] using the field of sets $\mathfrak{S}$ exists and is a linear functional on $C(X, A)$ (see also Smulian [14]).

4. 7. LEMMA. *There exists a function* $\gamma_0 \in \Gamma$ *such that* $N$ *is the set of zeros of* $\int_X f(t, \gamma_0(t))\, d\mu$ *for* $f \in C(X, A)$.

We argue as in [18, p. 154]. First we show that the sets in $\Delta$ have the finite intersection property. Let $\Gamma_i = \Gamma(f_i, \epsilon)$, $i = 1, 2$ be in $\Delta$. Let $h(\cdot)$ be the function related to the $f_i(\cdot)$ by Lemma 4. 6. It is easy to see that, for each $\gamma \in \Gamma$, $\mathfrak{A}(h, \gamma, \epsilon) = \bigcap \mathfrak{A}(f_i, \gamma, \epsilon)$. Since $h \in N$, there is a $\gamma \in \Gamma$ such that $\mathfrak{A}(h, \gamma, \epsilon)$ contains a set in $\mathfrak{A}(N)$ (see Lemma 4. 1). This set is contained in $\mathfrak{A}(f_i, \gamma, \epsilon)$ for $i = 1, 2$. Thus $\gamma \in \Gamma_1 \bigcap \Gamma_2$. Hence $\Delta$ is contained in a maximal family $\Delta'$ of subsets of $\Gamma$ with the finite intersection property. Letting $\nu$ be a set function defined for all subsets of $\Gamma$ to be one for sets in $\Delta'$ and zero otherwise, $\nu$ is additive. The arguments of [18, p. 155] may then be applied to give the desired results.

4. 8. LEMMA. *For each maximal ideal* $N$ *of* $C(X, A)$, $\mathfrak{A}(N)$ *is contained in one and only one maximal additive ideal of* $\mathfrak{L}$.

Let $\mathfrak{B}$ be such a m. a. ideal and $\mu$ be the associated $0-1$ measure function. If $\mathfrak{B}_1$ is another such m. a. ideal and $\mu_1$ is its associated $0-1$ measure function then by Lemma 4. 7 and the fact that two linear functionals with the same set of zeros are identical we have

$$\int_X f(t, \gamma(t))\, d\mu \equiv \int_X f(t, \gamma_1(t))\, d\mu_1 \tag{4.3}$$

for appropriate $\gamma$ and $\gamma_1$ in $\Gamma$. Now since $\mathfrak{B} \neq \mathfrak{B}_1$, by Wallman's criterion for a m. a. ideal there exists a set $E \in \mathfrak{B}$ which is disjoint with a set $E_1 \in \mathfrak{B}_1$. Since $E$ and $E_1$ are closed and $X$ is normal there is a bounded continuous real-valued function $f$ defined on $X$ which has the value one for $t \in E$, zero for $t \in E_1$. Defining $g(\cdot) \in C(X, A)$ by $g(t) = f(t)e$ ($e$ being the unit of $A$) we have from (4. 3) using $g(\cdot)$ that $1 = 0$ which is impossible. Hence $\mathfrak{B} = \mathfrak{B}_1$.

On the basis of the argument of Lemma 4.8 we may say there is just one $0-1$ measure function $\mu$ associated with the maximal ideal $N$ of $C(X, A)$ such that $N$ is the set of zeros of a multiplicative linear functional of the form (4.2).

4.9. LEMMA. *Let $N$ be a maximal ideal of $C(X, A)$ and let $\mu$ be the associated $0-1$ measure function. Then the collection $\Sigma(N)$ of all $\gamma \in \Gamma$ for which $N$ is the set of zeros of $\int_X f(t, \gamma(t))\, d\mu$ in $C(X, A)$ is non-empty and is $\bigcap \Gamma(f, \epsilon)$, $f \in N$.*

That the set $\Sigma(N)$ in question is non-empty follows from Lemma 4.7. Suppose $\gamma \in \bigcap \Gamma(f, \epsilon)$, $f \in N$, which we note is independent of the choice of $\epsilon > 0$. Then we have

$$\left| \int_X f(t, \gamma(t))\, d\mu \right| \leqq \int_X |f(t, \gamma(t))|\, d\mu \leqq \epsilon$$

for each $\epsilon > 0$, $f \in N$, and hence $\gamma \in \Sigma(N)$. Conversely if $\gamma \in \Sigma(N)$, then for each $f(\cdot) \in N$ there is by the arguments used for Lemma 4.6 (see [18, Lemma 2.4]) a function $g(\cdot) \in N$ such that $g(t, M) = |f(t, M)|$ for each $t \in X$, $M \in \mathfrak{M}$. Since

$$\int_X g(t, \gamma(t))\, d\mu = \int_X |f(t, \gamma(t))|\, d\mu = 0$$

it follows that for each $\epsilon > 0$, $\gamma \in \Gamma(f, \epsilon)$. Since $f(\cdot)$ is arbitrary in $N$, $\gamma \in \bigcap \Gamma(f, \epsilon)$.

4.10. THEOREM. *To each maximal additive ideal $\mathfrak{B}$ in $\mathfrak{L}$ there corresponds a decomposition of $\Gamma$ into disjoint sets $\Gamma(\mathfrak{B})$ such that the maximal ideals of $C(X, A)$ are in a 1-1 correspondence with the pairs $(\mathfrak{B}, \delta)$, $\delta$ being in $\Gamma(\mathfrak{B})$. This correspondence has the property that if $\delta'$ is any element of $\delta$ and if $\mu$ is the 0-1 measure function associated with $\mathfrak{B}$, then the maximal ideal which corresponds to $(\mathfrak{B}, \delta)$ is the set of $f(\cdot)$ for which*

$$\int_X f(t, \delta'(t))\, d\mu = 0. \tag{4.4}$$

Let $\mathfrak{B}$ be a m. a. ideal of $\mathfrak{L}$ and $\mu$ be its associated 0-1 measure function and let $\gamma$ and $\gamma_1$ be in $\Gamma$. We say that $\gamma \sim \gamma_1$ (with respect to $\mathfrak{B}$) if the relation (4.3) holds (with $\mu_1 = \mu$). Clearly $\sim$ is an equivalence relation defined on $\Gamma$ and provides a decomposition $\Gamma(\mathfrak{B})$ of $\Gamma$. If $\delta$ is such an equivalence class and $\gamma$ and $\gamma_1$ are in $\delta$, then the zeros of the integrals of (4.3) constitute the same maximal ideal $N$. That a maximal ideal is so obtained follows from the fact that the integral is a multiplicative linear functional on $C(X, A)$. Then by Lemma 4.9, $\delta = \bigcap \Gamma(f, \epsilon)$, $f \in N$.

Given the maximal ideal $N$ of $C(X, A)$ we correspond to it the pair $(\mathfrak{B}, \delta)$ where $\mathfrak{B}$ is the sole m. a. ideal of $\mathfrak{L}$ which contains $\mathfrak{A}(N)$ by Lemma 4. 3 and $\delta = \bigcap \Gamma(f, \epsilon)$, $f \in N$. $N$ is the set of zeros of (4. 4) by Lemma 4. 9. No two distinct maximal ideals can correspond to the same pair $(\mathfrak{B}, \delta)$. Also if $(\mathfrak{B}, \delta)$ is given it is clear from the above that there is precisely one maximal ideal $N$ of $C(X, A)$ which maps into $(\mathfrak{B}, \delta)$ under the correspondence.

We seek a criterion not involving a knowledge of the maximal ideals of $C(X, A)$ to determine whether or not $\gamma \sim \gamma_1$ (with respect to $\mathfrak{B}$). This we are able to do under the additional assumption that $X \times \mathfrak{M}$ is normal.

4. 11. LEMMA. *Let $X \times \mathfrak{M}$ be normal (as well as $X$). Then $\gamma_1 \sim \gamma_2$ (with respect to $\mathfrak{B}$) if and only if the closures of the sets $\{(t, \gamma_i(t)) \mid t \in F\}$, $i = 1, 2$ in $X \times \mathfrak{M}$ are not disjoint for each $F \in \mathfrak{B}$.*

Suppose that $\gamma_1 \nsim \gamma_2$. Then there exists $f(\cdot) \in C(X, A)$ for which

$$\int_X f(t, \gamma_i(t))\, d\mu = \begin{cases} 0, & i = 1 \\ 1, & i = 2 \end{cases}. \tag{4. 5}$$

Let $f'(\cdot)$ be the adjoint of $f(\cdot)$, $f'(t, M) \equiv \overline{f(t, M)}$. Then by the facts that the integral is a multiplicative linear functional and that the integrals for $f'(\cdot)$ have values conjugate to those for $f(\cdot)$, it follows that the relations (4. 5) hold for $g(\cdot) = f(\cdot) f'(\cdot)$ in place of $f(\cdot)$. Note that for each $(t, M)$, $g(t, M) \geqq 0$. Then by the continuity of the integrand as a function of $t$, its positivity and the fact that the integral is zero we see that

$F_1 = \{t \in X \mid |g(t, \gamma_1(t))| \leqq 1/4\}$ is a closed set and $\mu(F_1) = 1$.

By the same reasoning, the set $F_2 = \{t \in X \mid |g(t, \gamma_2(t))| \geqq 1/2\}$ is a closed set, $\mu(F_2) = 1$.

Let $F = F_1 \cap F_2$. Since each $\mu(F_i) = 1$ and $\mu$ is a 0-1 measure function on $\mathfrak{S}$, by Lemma 4. 4, $\mu(F) = 1$ and $F$ intersects every set in $\mathfrak{B}$. Thus $F \in \mathfrak{B}$ and is non-empty.

Consider the sets $S_i = \{(t, \gamma_i(t)) \mid t \in F_1 \cap F_2\}$, $i = 1, 2$. If $(t_0, M_0)$ is in the intersection of their closures in $X \times \mathfrak{M}$ it is readily verified by Lemma 2. 2 that $|g(t_0, M_0)| \leqq 1/4$, $|g(t_0, M_0)| \geqq 1/2$ which is a contradiction.

For the converse we suppose that there exists a set $F \in \mathfrak{B}$, such that the closures in $X \times \mathfrak{M}$ of the sets $S_i = \{(t, \gamma_i(t)) \mid t \in F\}$, $i = 1, 2$ are disjoint. Since $X \times \mathfrak{M}$ is normal, there is a bounded positive continuous function on $X \times \mathfrak{M}$ which has the value zero for $S_1$, one for $S_2$. Let $f(\cdot)$ be the corresponding member of $C(X, A)$ by Theorem 3. 1. Then the relations (4. 5) hold since $\mu(F) = 1$. This shows that $\gamma_1 \nsim \gamma_2$.

Thus we have liberated the equivalence relation $\sim$ (with respect to $\mathfrak{B}$) from the maximal ideals of $C(X, A)$ and have it described entirely in terms of $X$ and $\mathfrak{M}$. Next we start with the normal space $X$, and the compact space $\mathfrak{M}$ where $X \times \mathfrak{M}$ is normal and consider the $B^*$-algebra $C(X \times \mathfrak{M})$ of complex bounded continuous functions on $X \times \mathfrak{M}$. By Theorem 3.1 this is equivalent to $C(X, C(\mathfrak{M}))$. The equivalence classes of $\sim$ are families of functions defined on $X$ with values in $\mathfrak{M}$. To obtain the $\beta$-compactification of $X \times \mathfrak{M}$ we describe the topology of the space of maximal ideals of the latter algebra.

But first we note that if $(\mathfrak{B}, \delta)$ is a pair in Theorem 4.10 with the associated set function $\mu$, then if $f(\cdot)$ satisfies (4.4) so does its adjoint $f'(\cdot)$. Since the integral is a multiplicative functional and preserves complex conjugation, (4.4) holds for $f(\cdot)$ if and only if it holds for $f(\cdot)f'(\cdot)$. Thus $\mathfrak{A}(f, \gamma, \epsilon) \in \mathfrak{B}$ for each $\epsilon > 0$, $\gamma \in \delta$ is equivalent to (4.4) (see the discussion preceding Lemma 4.7 for the definition of $\mathfrak{A}(f, \gamma, \epsilon)$).

4.12. THEOREM. *Let $X$ and $X \times \mathfrak{M}$ be normal where $\mathfrak{M}$ is compact. The points of the $\beta$-compactification of $X \times \mathfrak{M}$ are in a 1-1 correspondence with the pairs $(\mathfrak{B}, \delta)$ where $\mathfrak{B}$ is a m. a. ideal of $\mathfrak{L}$ and $\delta$ is an equivalence class of $\sim$ with respect to $\mathfrak{B}$. Let $E$ be any set in the $\beta$-compactification. Then its closure $\bar{E}$ there is the set of all $(\mathfrak{B}_1, \delta_1)$ for which $\mathfrak{A}(f, \gamma, \epsilon) \in \mathfrak{B}_1$ for each $\epsilon > 0$, $\gamma_1 \in \delta_1$ whenever $\mathfrak{A}(f, \gamma, \epsilon) \in \mathfrak{B}$ for each $\epsilon > 0$, $(\mathfrak{B}, \delta) \in E$, $\gamma \in \delta$.*

Since the maximal ideals are given as the zeros of corresponding Hildebrandt integrals (Theorem 4.10), by the previous discussion this is a description of closure in the space of maximal ideals of $C(X, C(\mathfrak{M}))$ using the Stone topology (see § 2). Thus we have succeeded in describing the $\beta$-compactification in terms of the $\beta$-compactification of $X$, $P_{t \in X}\mathfrak{M}$, continuous functions $X \times \mathfrak{M}$ and closure in $X \times \mathfrak{M}$. That the m. a. ideals of $\mathfrak{L}$ constitute the $\beta$-compactification of $X$ is assured since $X$ is normal.

Concerning the relation of $(\mathfrak{B}_1, \delta_1) \in \bar{E}$ to $E$ we can say a bit more. Let $\mathfrak{E}$ be the additive ideal of the lattice $\mathfrak{L}$ which consists of all closed subsets of $\mathfrak{L}$ common to all the $\mathfrak{B}$ of $(\mathfrak{B}, \delta)$ in $E$. $\mathfrak{E}$ is non-empty as it contains at least the set $X$.

4.14. THEOREM. *In the notation of Theorem* 4.13, *if $(\mathfrak{B}_1, \delta_1) \in \bar{E}$, then $\mathfrak{E}$ is contained in $\mathfrak{B}_1$.*

Suppose that there exists a set $F$ in $\mathfrak{E}$ which is not in $\mathfrak{B}_1$. Since $\mathfrak{B}_1$ is a m. a. ideal, there is a set $F_1 \in \mathfrak{B}_1$ which is disjoint with $F$. By the normality of $X$, there is a continuous bounded real-valued function $f(t)$ which is one on $F_1$ and zero on $F$. Then let $g(t) = f(t)e$. We have

$$\int_X g(t, \gamma(t))\, d\mu = \int_F g(t, \gamma(t))\, d\mu = 0$$

for each $\mu$ associated with $\mathfrak{B}$ of $(\mathfrak{B}, \delta) \in E$, $\gamma \in \delta$. Also the integral for $\mu_1$ associated with $B_1$ and $\gamma_1 \in \delta_1$ has the value 1. This is impossible as $(B_1, \delta_1) \in \bar{E}$.

**5. $X$ compact, $A$ a Banach algebra.** In this section we are concerned with the space of maximal ideals of $C(X, A)$ when $X$ is a compact Hausdorff space and $A$ is a Banach algebra but not a $B^*$-algebra. In particular it is shown that the space of maximal ideals is homeomorphic to $X$ if $A$ is a primary ring in the sense of Šilov [13], that is, possesses one maximal ideal.

5. 1. LEMMA. *Let $X$ be a compact Hausdorff space and every maximal ideal in $C(X, A)$ be fixed. Then the space of maximal ideals $\mathfrak{N}$ of $C(X, A)$ is homeomorphic to $X \times \mathfrak{M}$.*

It is easy to verify that if $t_i \in X$, $M_i \in \mathfrak{M}$, $i = 1, 2$, then the maximal ideals $\{f(\cdot) \mid f(t_i) \in M_i\}$, $i = 1, 2$ are the same if and only if $t_1 = t_2$ and $M_1 = M_2$. Thus $\mathfrak{N}$ may be considered as the set of all pairs $(t, M)$, $t \in X$, $M \in \mathfrak{M}$ supplied with an appropriate topology. But by Lemma 2. 2, for each $f(\cdot) \in C(X, A)$, $f(t, M)$ is continuous on $X \times \mathfrak{M}$ which is a compact Hausdorff space. By a theorem of Gelfand [3], $\mathfrak{N}$ is homeomorphic to $X \times \mathfrak{M}$.

5. 2. THEOREM. *Let $X$ be a compact Hausdorff space and $A$ be a primary ring. Then the space $\mathfrak{N}$ of maximal ideals of $C(X, A)$ is homeomorphic to $X$.*

By Lemma 5. 1 this result holds if every maximal ideal $N \in \mathfrak{N}$ is fixed. Suppose that $N$ is not fixed. Let $M$ be the maximal ideal of $A$. Then for each $t_0 \in X$ there exists $f(\cdot) \in N$ such that $f(t_0, M) \neq 0$. Since $f(t, M)$ is continuous in $t$ by Lemma 2. 2, there exists a neighborhood $U(t_0)$ of $t_0$ in $X$ where $f(t, M) \neq 0$. A finite number of these neighborhoods suffice to cover $X$. Let $f_1(\cdot), \cdots, f_n(\cdot)$ be the corresponding elements of $N$. We define $g_i(\cdot)$ by $g_i(t) = \overline{f_i(t, M)}e$ for each $t \in X$, $i = 1, \cdots, n$. Then $g_i(\cdot) \in C(X, A)$, $i = 1, \cdots, n$. Set $h(\cdot) = \sum_{i=1}^{n} f_i(\cdot) g_i(\cdot)$. It is easy to see that for each $t \in X$, $h(t, M) \neq 0$ and that $h(\cdot) \in N$. For each $t \in X$, $[h(t)]^{-1}$ exists in $A$ ([3]). But $[h(t)]^{-1}$ as a function of $t$ is a continuous function of a continuous function defined on $X$ ([9, Theorem 5. 2. 3]) and moreover is bounded since $X$ is compact. Thus $h(\cdot)$ has an inverse in $C(X, A)$ which contradicts the fact that $h(\cdot)$ is in the maximal ideal $N$.

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# A FORM OF GREEN'S TRANSFORMATION.*

By C. TRUESDELL.

The transformation of a volume integral into a surface integral familiar in the equivalent forms

$$(1)\qquad \oint_S d\boldsymbol{S}\boldsymbol{b} = \oint_V \operatorname{grad} \boldsymbol{b}\, d\mathfrak{V},\quad \oint_S d\boldsymbol{S}\cdot\boldsymbol{b} = \oint_V \operatorname{div} \boldsymbol{b}\, d\mathfrak{V},$$
$$\oint_S d\boldsymbol{S}\times\boldsymbol{b} = \oint_V \operatorname{curl} \boldsymbol{b}\, d\mathfrak{V},$$

which is known generally as "Gauss's theorem," "Ostrogradski's theorem," "Green's theorem," or the "divergence theorem," was first employed in a special case by Lagrange,[1] and its systematic use in mathematical physics was initiated by Green,[2] who discovered the equivalent forms

$$(2)\qquad \oint_S d\boldsymbol{S}\cdot(\phi \operatorname{grad}\psi - \psi \operatorname{grad}\phi) = \oint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, d\mathfrak{V},$$
$$\oint_S d\boldsymbol{S}\cdot\phi \operatorname{grad}\psi = -\oint_V (\phi\nabla^2\psi + \operatorname{grad}\phi\cdot\operatorname{grad}\psi)\, d\mathfrak{V}.$$

There are many other forms of this transformation. Any form containing at least as many arbitrary functions of position as the order of the highest derivative occurring in it is equivalent to all the others, and no new results can be obtained by building up apparently more complicated combinations. Some of these, however, are useful in mechanics and physics, so that various books and papers pursue the manipulations necessary to obtain them. The purpose of the present note is to give an economical derivation of a new form sufficiently elaborate to include and relate a number of known examples.

For the polyadic $n$-th power[3] of a vector $\boldsymbol{b}$ let us introduce the notation $\boldsymbol{b}^{(n)}$:

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[1] J. L. Lagrange, *Œuvres*, vol. 1, pp. 151-316; see § 45. In the work of Gauss also only special cases appear: *Gesammelte Werke*, vol. 5, pp. 3-22, see§§ 3-5, and pp. 195-242, see § 22.

[2] G. Green, *Mathematical Papers*, pp. 23-26.

[3] Our polyadic notation is essentially that of Gibbs and Wilson, *Vector Analysis*, New Haven (1902). In indicial notation $\boldsymbol{b}^{(n)}$ is $b^{i_1}b^{i_2}\cdots b^{i_n}$.

$$(3)\qquad \boldsymbol{b}^{(-1)} \equiv 0,\quad \boldsymbol{b}^{(0)} \equiv 1,\quad \boldsymbol{b}^{(1)} \equiv \boldsymbol{b},\quad \boldsymbol{b}^{(2)} \equiv \boldsymbol{bb},\quad \boldsymbol{b}^{(3)} \equiv \boldsymbol{bbb}, \cdots.$$

Let $\boldsymbol{\Sigma}$ be any polyadic (i. e., tensor), and let the notation $\{\boldsymbol{\Sigma b}^{(n)}\}$ stand for the polyadic [4]

$$(4)\qquad \{\boldsymbol{\Sigma b}^{(n)}\} \equiv \boldsymbol{b}^{(n)}\boldsymbol{\Sigma} + \boldsymbol{b}^{(n-1)}\boldsymbol{\Sigma b} + \boldsymbol{b}^{(n-2)}\boldsymbol{\Sigma b}^{(2)} + \cdots + \boldsymbol{\Sigma b}^{(n)},$$

which is completely symmetrical with respect to $\boldsymbol{b}$.

Let $\boldsymbol{b}$ and $\boldsymbol{c}$ be arbitrary vector fields, and let $\boldsymbol{r}$ be the radius vector. First, since $\mathbf{grad}\,\boldsymbol{r} = \boldsymbol{I}$, we have

$$(5)\qquad \operatorname{div}(\boldsymbol{bcr}^{(n)}) = \boldsymbol{c}\{\boldsymbol{br}^{(n-1)}\} + (\boldsymbol{b}\cdot\mathbf{grad}\,\boldsymbol{c})\boldsymbol{r}^{(n)} + (\boldsymbol{c}\operatorname{div}\boldsymbol{b})\boldsymbol{r}^{(n)}.$$

Similarly

$$(6)\qquad \mathbf{grad}(\boldsymbol{b}\cdot\boldsymbol{cr}^{(n)}) = \boldsymbol{b}\cdot\boldsymbol{c}\{\boldsymbol{r}^{(n-1)}\boldsymbol{I}\} + (\mathbf{grad}\,\boldsymbol{c}\cdot\boldsymbol{b} + \mathbf{grad}\,\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{r}^{(n)}.$$

Since

$$(7)\qquad \boldsymbol{b}\cdot\mathbf{grad}\,\boldsymbol{c} - \mathbf{grad}\,\boldsymbol{c}\cdot\boldsymbol{b} = \boldsymbol{curl}\,\boldsymbol{c}\times\boldsymbol{b},$$

from (5) and (6) we obtain

$$\begin{aligned}(8)\qquad &\operatorname{div}[(\boldsymbol{bc}+\boldsymbol{cb})\boldsymbol{r}^{(n)}] - \mathbf{grad}(\boldsymbol{b}\cdot\boldsymbol{cr}^{(n)})\\ &\quad = \boldsymbol{c}\{\boldsymbol{br}^{(n-1)}\} + \boldsymbol{b}\{\boldsymbol{cr}^{(n-1)}\} - \boldsymbol{b}\cdot\boldsymbol{c}\{\boldsymbol{r}^{(n-1)}\boldsymbol{I}\}\\ &\qquad + (\mathbf{curl}\,\boldsymbol{c}\times\boldsymbol{b} + \mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{b}\operatorname{div}\boldsymbol{c} + \boldsymbol{c}\operatorname{div}\boldsymbol{b})\boldsymbol{r}^{(n)}.\end{aligned}$$

Now the familiar forms (1) remain valid when $\boldsymbol{b}$ is replaced by a polyadic (tensor) $\boldsymbol{\Sigma}$ of any rank. Hence integrating (8) over a volume yields

$$\begin{aligned}(9)\qquad &\oint_S [d\boldsymbol{S}\cdot(\boldsymbol{bc}+\boldsymbol{cb})\boldsymbol{r}^{(n)} - d\boldsymbol{S}\boldsymbol{b}\cdot\boldsymbol{cr}^{(n)}]\\ &\quad = \int_V [\boldsymbol{b}\{\boldsymbol{cr}^{(n-1)}\} + \boldsymbol{c}\{\boldsymbol{br}^{(n-1)}\} - \boldsymbol{b}\cdot\boldsymbol{c}\{\boldsymbol{r}^{(n-1)}\boldsymbol{I}\}\\ &\qquad + (\mathbf{curl}\,\boldsymbol{c}\times\boldsymbol{b} + \mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{b}\operatorname{div}\boldsymbol{c} + \boldsymbol{c}\operatorname{div}\boldsymbol{b})\boldsymbol{r}^{(n)}]\,d\mathfrak{V},\end{aligned}$$

which is the desired result.

By putting $n = 0$ we obtain the transformation

$$\begin{aligned}(10)\qquad &\oint_S [d\boldsymbol{S}\cdot(\boldsymbol{bc}+\boldsymbol{cb}) - d\boldsymbol{S}\,\boldsymbol{b}\cdot\boldsymbol{c}]\\ &\quad = \int_V [\mathbf{curl}\,\boldsymbol{c}\times\boldsymbol{b} + \mathbf{curl}\,\boldsymbol{b}\times\boldsymbol{c} + \boldsymbol{b}\operatorname{div}\boldsymbol{c} + \boldsymbol{c}\operatorname{div}\boldsymbol{b}]\,d\mathfrak{V},\end{aligned}$$

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[4] In indicial notation $\{\boldsymbol{\Sigma b}^{(n)}\}$ is $\sum_{k=0}^{n}[(\prod_{j=1}^{k} b^{i_j})\Sigma^{i_1\ldots i_n}(\prod_{j=k+1}^{n} b^{i_j})]$.

discovered by Burgatti,[5] who showed it to contain as special cases a formula of Levi-Civita in the theory of liquid jets, a formula of Teofilato in aerodynamics and a formula of Vaschy in electromagnetic theory, as well as various classical formulae of potential theory. Earlier, however, Lamb [6] and J. J. Thompson [7] in their studies of the rotational motions of fluids had derived the special case when $\boldsymbol{c} = \boldsymbol{b} =$ the velocity of a motion of an incompressible substance ($\operatorname{div} \boldsymbol{b} = 0$); Lamb's result is that if $\lim r^2 b = 0$ as $r \to \infty$ then the integral $\int_v \mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{b}\, d\mathfrak{V}$ taken over the whole of a continuous motion filling all space must vanish.

By putting $n = 1$ in our general identity (9) we obtain

$$\oint_s [d\boldsymbol{S} \cdot (\boldsymbol{bc} + \boldsymbol{cb})\boldsymbol{r} - d\boldsymbol{S}\boldsymbol{b} \cdot \boldsymbol{cr}]$$

$$= \int_v [\boldsymbol{bc} + \boldsymbol{cb} - \boldsymbol{b} \cdot \boldsymbol{cI} + (\mathbf{curl}\, \boldsymbol{c} \times \boldsymbol{b} + \mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{b} \operatorname{div} \boldsymbol{c} + \boldsymbol{c} \operatorname{div} \boldsymbol{b})\boldsymbol{r}]\, d\mathfrak{V}. \tag{11}$$

Taking the scalar of this dyadic equation [8] yields

$$\oint_s [d\boldsymbol{S} \cdot (\boldsymbol{bc} + \boldsymbol{cb}) \cdot \boldsymbol{r} - d\boldsymbol{S} \cdot \boldsymbol{rb} \cdot \boldsymbol{c}]$$

$$= \int_v [-\boldsymbol{b} \cdot \boldsymbol{c} + \boldsymbol{r} \cdot (\mathbf{curl}\, \boldsymbol{c} \times \boldsymbol{b} + \mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{b} \operatorname{div} \boldsymbol{c} + \boldsymbol{c} \operatorname{div} \boldsymbol{b})]\, d\mathfrak{V}. \tag{12}$$

Put $\boldsymbol{c} = \boldsymbol{b}$ and suppose $\operatorname{div} \boldsymbol{b} = 0$; then

$$\tfrac{1}{2} \int_v b^2 d\mathfrak{V} = \oint_s d\boldsymbol{S} \cdot (\tfrac{1}{2}\boldsymbol{I}b^2 - \boldsymbol{bb}) \cdot \boldsymbol{r} + \int_v \boldsymbol{r} \cdot \mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{b}\, d\mathfrak{V}. \tag{13}$$

When $\boldsymbol{b}$ is the velocity vector of a motion, $\mathbf{curl}\, \boldsymbol{b}$ is the vorticity; for an incompressible substance, since $\operatorname{div} \boldsymbol{b} = 0$ we may apply (13), and if further the susbtance be of uniform density, the left side is proportional to the total kinetic energy, and we obtain another result of Lamb [9] and J. J. Thompson.[10]

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[5] P. Burgatti, "Intorno a una formula generale di trasformazione di un integrale di spazio in uno di superficie e alle sue varie deduzioni," *Bolletino della Unione Matematica Italiana*, vol. 10 (1931), pp. 1-5.

[6] H. Lamb, *A Treatise on the Mathematical Theory of the Motion of Fluids*, Cambridge (1879), see § 136. Also *Hydrodynamics*, 6th ed., Cambridge (1932), § 153.

[7] J. J. Thomson, *A Treatise on the Motion of Vortex Rings*, London (1883). See § 4.

[8] In tensorial language, contracting the equation $\Phi^{ij} = \cdot\ \cdot\ \cdot$; operationally, inserting a dot into each dyad of the dyadic.

[9] *Loc. cit.*

[10] *Op. cit.*, § 6.

From this identity we notice also that if $\mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{b} = 0$ everywhere and if $\boldsymbol{b} = 0$ upon the boundary surface $\mathcal{S}$, then both integrals on the right side vanish and hence $\boldsymbol{b} = 0$. Thus we have an important theorem stated in principle but not adequately proved by Duhem:[11] There can exist no continuously differentiable motion of an incompressible substance such that (i) the vorticity is everywhere zero or parallel to the velocity, (ii) the material adheres to all finite boundaries without slipping, and in infinite regions $\lim r^3 b^2 = 0$ as $r \to \infty$. For a second application, put $\boldsymbol{b} = \mathbf{curl}\, \boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity vector of a motion of any substance, so that $\operatorname{div} \boldsymbol{b} = 0$ again. From (13) we deduce the following result: In any motion such that the vorticity is everywhere parallel to its own curl, if the vorticity $\boldsymbol{b}$ vanish upon all finite boundaries and $\lim r^3 b^2 = 0$ as $r \to \infty$ in all infinite regions, the vorticity itself must vanish everywhere. This theorem is interesting because the viscous forces in a liquid of uniform viscosity depend upon $\mathbf{curl}\, \boldsymbol{b}$ only, so that when $\mathbf{curl}\, \boldsymbol{b} = 0$ even if $\boldsymbol{b} \neq 0$, the equations of motion reduce to those of perfect fluids; our result shows that such simple motions cannot be imbedded continuously in a region whose boundaries are in irrotational motion. Returning to the dyadic identity (11), we may obtain another form by taking its vector or cross[12], viz.,

$$\oint_S [d\boldsymbol{S} \cdot (\boldsymbol{bc} + \boldsymbol{cb}) \times \boldsymbol{r} - d\boldsymbol{S} \times \boldsymbol{rb} \cdot \boldsymbol{b}]$$

$$= \int_V (\mathbf{curl}\, \boldsymbol{c} \times \boldsymbol{b} + \mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{c} + \boldsymbol{b} \operatorname{div} \boldsymbol{c} + \boldsymbol{c} \operatorname{div} \boldsymbol{b}) \times \boldsymbol{r}\, d\mathfrak{V}. \tag{14}$$

Again put $\boldsymbol{c} = \boldsymbol{b}$ and suppose $\operatorname{div} \boldsymbol{b} = 0$; the resulting special case is

$$\oint_S [d\boldsymbol{S} \cdot \boldsymbol{br} \times \boldsymbol{b} + \tfrac{1}{2} b^2 d\boldsymbol{S} \times \boldsymbol{r}] = \int_V \boldsymbol{r} \times (\mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{b})\, d\mathfrak{V}, \tag{15}$$

an identity used by Poincaré[13] in his analysis of the vorticity of fluid motions.

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[11] P. Duhem, "Recherches sur l'hydrodynamique, 5e partie, Le théorème de Lagrange et les conditions aux limites," *Annales de la Faculté des Sciences de Toulouse*, ser. 2, vol. 5 (1903), pp. 353-376; see Chapter II, § 1. Duhem considered only the case $\mathbf{curl}\, \boldsymbol{b} = 0$. The more general result for "Beltrami motions" ($\mathbf{curl}\, \boldsymbol{b} \times \boldsymbol{b} = 0$) is here stated and proved for the first time. Duhem's theorem is not to be confused with a somewhat similar result in potential theory, where the condition $\boldsymbol{b} = 0$ upon $\mathcal{S}$ is not required, but the domain must be simply connected.

[12] "Taking the cross" of $\Phi^{ij} = \ldots$ means forming $\epsilon_{ijk}\Phi^{jk} = \ldots$; operationally, inserting a cross into each dyad of the dyadic.

[13] H. Poincaré, *Théorie des tourbillons*, Paris (1893), § 115.

The occurrence of $\boldsymbol{r}^{(n)}$ in the general transformation (9) suggests that the vectorial moments of a vector field, particularly of a solenoidal vector field, have simple and useful properties. Particular cases, such as polyadic cross moments $\boldsymbol{r} \times (\boldsymbol{r} \times (\boldsymbol{r} \times \cdots (\boldsymbol{r} \times \boldsymbol{\Sigma}) \cdots))$ and special types of scalar moments like $\boldsymbol{r} \cdot (\boldsymbol{r} \times (\boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{\Sigma})))$ for a dyadic $\boldsymbol{\Sigma}$, suggest themselves at once. In a later paper [14] I shall show that the study of vectorial moments enables the method introduced by Poincaré in the theory of vortices to be generalized so as to apply to any deformable continuum in motion.

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# DIFFERENTIAL EQUATIONS REFERRED TO A VARIABLE METRIC.*

By D. C. Lewis.

**1. Introduction.** In a previous paper,[1] hereafter denoted by MP, the author studied the dependence of solutions of systems of differential equations on the initial conditions with the help of a general Finsler metric, which, of course, includes as a special case a Riemann metric. In applications to exterior ballistics and other problems, the success of this method is naturally seen to depend on the choice of the metric, but it was disappointing that for *no* choice of the metric was it possible to deduce from this theory certain well known results about even linear differential equations. For example, it appeared to be impossible to deduce by this method that any solution of the Bessel differential equation is $O(t^{-\frac{1}{2}})$, where $t$ is the independent variable. It thus was felt that the method as so far developed was intrinsically too crude to yield the desired results in applications.

The present paper makes at least a first step in remedying this situation. This is done by introducing a metric which varies not only from point to point of the space, but also varies with the independent variable $t$ of the differential system. The resulting theorems seem to be but trivial modifications of those in MP, and yet the power of the method when applied to linear differential equations is greatly increased, as is shown in Section **3**, where some results of Wintner [2] and Leighton [3] are obtained by this method together with the well known result on Bessel functions previously alluded to.

The success of the method still depends on the choice of the metric, and a general theory of such a choice is still unavailable. Nevertheless, the possibility of approximate linearization in many notable cases indicates the essential importance of the fairly complete analysis of this question in the linear case, appearing in Section **5**.

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[1] D. C. Lewis, "Metric properties of differential equations," *American Journal of Mathematics*, vol. 71 (1949), pp. 294-312.

[2] A. Wintner, "Stability and high frequency," *Journal of Applied Physics*, vol. 18 (1947), pp. 941-942.

[3] W. Leighton, "Bound for the solutions of a second-order linear differential equation," *Proceedings of the National Academy of Sciences*, vol. 35 (1949), pp. 190-191.

**2. The fundamental theorems referred to a Finsler variable metric.** The idea of a variable metric is simply that the distance between two points is to be specified only when a variable $t$, which we may think of as representing time, takes on a definite value. If we are thinking of a Riemann metric $\sum_{ij} g_{ij}dx_idx_j$, this means that the $g$'s are not only functions of $x_1, \cdots, x_n$ but also are functions of $t$. It means that the "distance" between two points changes in general with $t$, even if the points themselves are fixed. In the immediate sequel we consider the somewhat more general case of a Finsler metric.

Let $f[x, \dot{x}, t] = f(x_1, \cdots, x_n, \dot{x}_1, \cdots, \dot{x}_n, t)$ be of class $C'$ in all of its $2n+1$ arguments and suppose that, for each fixed value of $t$, it is of class $C'''$ in the $x$'s and $\dot{x}$'s. Furthermore we assume $f[x, \dot{x}, t]$ to be positively homogeneous of degree $+1$ in $\dot{x}_1, \cdots, \dot{x}_n$, and that it is positive unless all the $\dot{x}$'s are zero. Finally, in order to insure the "regularity" of the variational problem $\delta \int f[x, \dot{x}, t]d\tau = 0$, we assume that $f_{\dot{x}_i\dot{x}_j}[x, \dot{x}, t]\lambda^i\lambda^j > 0$ for all sets $\lambda^1, \cdots, \lambda^n$ not proportional to $\dot{x}_1, \cdots, \dot{x}_n$. Here, as in the sequel, the summation convention of tensor analysis is used, with all repeated indices summed from 1 to $n$.

The length of an arc, $x_i = x_i(\tau)$, $i = 1, \cdots, n$, $0 \leqq \tau \leqq 1$, is defined *relative to each fixed value of* $t$ as the value of the integral,

$$\int_0^1 f[x(\tau), x'(\tau), t]d\tau,$$

which exists if the arc is sufficiently regular and is, moreover, independent of the parametrization on account of the assumed homogeneity of $f[x, \dot{x}, t]$. The *distance* between two points is the length of the shortest arc connecting the two points, and this also naturally will in general depend upon $t$ even though the two points are fixed when $t$ changes. The parametric equations of such a *geodesical* arc are well known to satisfy Euler's equations,

$$(2.1) \quad f_{x_i}[x(\tau), x'(\tau), t] - (d/d\tau)f_{\dot{x}_i}[x(\tau), x'(\tau), t] = 0 \quad i = 1, \cdots, n.$$

Concerning the system of differential equations,

$$(2.2) \qquad dx/dt = X^i(x_1, \cdots, x_n, t) = X^i[x, t], \qquad i = 1, \cdots, n,$$

we assume that the $X$ are of class $C'$. Our two fundamental theorems may be stated as follows:

THEOREM 1. *Let us consider two integral arcs of the system* (2.2).

$$C_0: x_i = x_i(t, 0) \text{ and } C_1: x_i = x_i(t, 1), \quad i = 1, \cdots, n, \ t_0 \leqq t \leqq t + h.$$

*We let $D(t)$ denote the distance relative to $t$ between $x(t,0)$ and $x(t,1)$ measured along the geodesical arc $g_t$. We furthermore assume that all these $g_t$ are included within an open region $R$, and that no complete geodesic obtained by extending a $g_t$ contains entirely within $R$ an arc whose end points are conjugate to each other* (in the sense of the calculus of variations).

*If*

$$(2.3)\quad \alpha(t) \leqq \{f_{x_i}[x,\lambda,t]X^i[x,t] + f_{\dot{x}_i}[x,\lambda,t]X^i{}_{x_j}[x,t]\lambda^j + f_t[x,\lambda,t]\} \leqq \beta(t)$$

*for every set $\lambda^1, \cdots, \lambda^n$ $(=\lambda^1(t), \cdots, \lambda^n(t))$ such that $f[x,\lambda,t]=1$ and at every point $[x]$ on $g_t$, then*

$$(2.4)\quad D(t_0)\exp\int_{t_0}^{t}\alpha(u)\,du \leqq D(t) \leqq D(t_0)\exp\int_{t_0}^{t}\beta(u)\,du, \qquad t_0 \leqq t \leqq +h.$$

THEOREM 2. *Let $C_0$ and $C_1$ be defined as in the preceding theorem. Let $\Gamma_{t_0}$ be an arbitrary sufficiently regular curve, with parametric equations, $x_i=\xi_i(\tau)$, $i=1,\cdots,n$, $0\leqq\tau\leqq 1$, whose end points are at $[x(t_0,0)]$ and $[x(t_0,1)]$ respectively. Let $C_\tau$ be the curve whose parametric equations, $x_i=x_i(t,\tau)$, satisfy the system (2.2) for each fixed value of $\tau$ on the unit interval and take on the initial values $x_i(t_0,\tau)=\xi_i(\tau)$. Let $\Gamma_t$ be the arc whose parametric equations, for each fixed $t$, $(t_0\leqq t\leqq t_0+h)$, are likewise $x_i=x_i(t,\tau)$, $0\leqq\tau\leqq 1$. Let $L(t)$ denote the length of $\Gamma_t$ relative to $t$.*

*If*

$$(2.3^*)\quad \alpha^*(t) \leqq \{f_{x_i}[x,\lambda,t]X^i[x,t] + f_{\dot{x}_i}[x,\lambda,t]X^i{}_{x_j}[x,t]\lambda^j + f_t[x,\lambda,t]\} \leqq \beta^*(t),$$

*for every set of $\lambda$'s such that $f[x,\lambda,t]=1$ and at every point $[x]$ of $\Gamma_t$, then*

$$(2.4^*)\quad L(t_0)\exp\int_{t_0}^{t}\alpha^*(u)\,du \leqq L(t) \leqq L(t_0)\exp\int_{t_0}^{t}\beta^*(u)\,du.$$

The proofs of these two theorems are obvious modifications of the proofs of the corresponding theorems in MP. Moreover Theorem 1 may be obtained as an easy consequence of Theorem 2 in the manner indicated on p. 299 of MP. For the sake of brevity we shall therefore prove here only Theorem 2. For this purpose, we observe that, by our definition of length relative to $t$,

$$(2.5)\qquad L(t) = \int_0^1 f[x(t,\tau), x_\tau(t,\tau), t]\,d\tau.$$

Differentiating under the integral sign, we have (after making use of (2.2))

$$(2.6)\quad L'(t) = \int_0^1 \{f_t[x(t,\tau), x_\tau(t,\tau), t] + f_{x_i}[x(t,\tau), x_\tau(t,\tau), t]X^i[x(t,\tau), t] + f_{\dot{x}_i}[x(t,\tau), x_\tau(t,\tau), t]X^i{}_{x_j}[x(t,\tau), t]\partial x_j/\partial\tau\}d\tau.$$

We introduce the arc length $s$ as a new parameter defined by

$$(2.7)\qquad s = s(t,\tau) = \int_0^\tau f[x(t,\tau'), x_\tau(t,\tau'), t]d\tau',$$

which has the obvious properties

$$(2.8)\qquad \partial s/\partial\tau = f[x(t,\tau), x_\tau(t,\tau), t] > 0, \qquad s(t,1) = L(t).$$

Moreover, on account of the assumed homogeneity of $f[x, \dot{x}, t]$, the parameter in (2.7) can be changed from $\tau'$ to $s'$, where $s' = s(t,\tau')$, with the result, deduced from (2.7), that

$$s = \int_0^s f[x\{t,s'\}, x_s\{t,s'\}, t]ds',$$

where $x\{t, s(t,\tau)\} = x(t,\tau)$, so that

$$(2.9)\qquad f[x\{t,s\}, x_s\{t,s\}, t] = ds/ds = 1.$$

The integrand in (2.6) is positively homogeneous of degree $+1$ in $\dot{x}_1, \cdots, \dot{x}_n$, because $f[x, \dot{x}, t]$ is. Hence (2.6) may be written in the form

$$L'(t) = \int_0^{L(t)} \{f_t[x, x_s, t] + f_{x_i}[x, x_s, t]X^i[x, t] + f_{\dot{x}_i}[x, x_s, t]X^i{}_{x_j}[x, t](\partial x_j/\partial s)\}ds.$$

It now follows at once from (2.3*), by taking $\lambda^i = x^i{}_s$ in virtue of (2.9), that

$$\alpha^*(t)L(t) \leqq L'(t) \leqq \beta^*(t)L(t),$$

from which the proof of Theorem 2 is completed by a simple integration.

**3. Special case of a Riemann metric. Second order linear equations.** Of particular importance is the case when

$$(3.1)\qquad f[x, \dot{x}, t] = (g_{ij}[x, t]\dot{x}_i\dot{x}_j)^{\frac{1}{2}}.$$

The middle member of (2.3), or (2.3*), we now verify can be written as follows:

$$(3.2)\qquad Q^*[\lambda] = (\tfrac{1}{2}X^k\partial g_{ij}/\partial x_k + \tfrac{1}{2}\partial g_{ij}/\partial t + g_{ik}X^k{}_{x_j})\lambda^i\lambda^j.$$

In obtaining this result, we of course make use of the fact that $\lambda^1, \cdots, \lambda^n$ is a unit vector, i. e. $g_{ij}\lambda^i\lambda^j = 1$. The $\alpha(t)$ and $\beta(t)$ of the previous section are therefore lower and opper bounds respectively of the quadratic form $Q^*[\lambda]$ for values of the $\lambda$'s which are components of unit vectors.

The power of our fundamental theorems of the previous section, when a Riemann metric is employed, can be illustrated in the case of second order linear systems as follows:

Consider the second order linear differential equation

$$(px')' + qpx' + rx = 0 \tag{3.3}$$

in which the primes denote differentiation with respect to the independent variable $t$, and in which $p, q, r$, $(pr)'$, $(pq)'$ are continuous functions of $t$ for $t_0 \leqq t < T$. By setting $x_1 = x$ and $x_2 = px'$, we may write the equivalent system in the form,

$$dx_1/dt = p^{-1}x_2 = X^1, \qquad dx_2/dt = -rx_1 - qx_2 = X^2, \tag{3.4}$$

which can evidently be considered as a special case of (2.2) with $n = 2$. With this system, assuming $pr - (1/4)(pq)^2 > 0$, we consider the Riemann metric

$$g_{11} = pr, \qquad g_{12} = g_{21} = \tfrac{1}{2}pq, \qquad g_{22} = 1, \tag{3.5}$$

which, after a simple calculation and upon setting $\lambda^1 = \lambda$, $\lambda^2 = \mu$, allows us to specialize (3.2) in the form,

$$Q^*[\lambda] = \tfrac{1}{2}q\{pr\lambda^2 + pq\lambda\mu + \mu^2\} + \tfrac{1}{2}\{(pr'\lambda^2 + (pq)'\lambda\mu\}$$

Since $(\lambda, \mu)$, as a unit vector, must satisfy the relation $pr\lambda^2 + pq\lambda\mu + \mu^2 = 1$, it is clear that

$$Q^*[\lambda] = -\tfrac{1}{2}q + \tfrac{1}{2}(pr)'\lambda^2 + \tfrac{1}{2}(pq)'\lambda\mu. \tag{3.6}$$

Moreover, from the identities,

$$\begin{aligned} pr\lambda^2 + pq\lambda\mu + \mu^2 &\equiv (pr - \tfrac{1}{4}p^2q^2)\lambda^2 + (\tfrac{1}{2}pq\lambda + \mu)^2 \\ &\equiv ([pr]^{\frac{1}{2}}\lambda \pm \mu)^2 + (pq \mp 2[pr]^{\frac{1}{2}})\lambda\mu, \end{aligned}$$

it is readily deduced that our unit vector components must satisfy the inequalities,

$$\lambda^2 \leqq (pr - \tfrac{1}{4}p^2q^2)^{-1}, \tag{3.7}$$

$$-(2[pr]^{\frac{1}{2}} - pq)^{-1} \leqq \lambda\mu \leqq (2[pr]^{\frac{1}{2}} + pq)^{-1}. \tag{3.8}$$

Thus from (3.6), (3.7), and (3.8), it is very easy to obtain explicit lower and upper bounds for $Q[\lambda]$. Let us consider somewhat further two special cases:

*Case* 1. $q \equiv 0$, $p > 0$, $r > 0$, and $(pr)' \geqq 0$. This is the self-adjoint case considered for instance by Wintner [2] and Leighton.[3]

Here we clearly have $\alpha(t) \equiv 0 \leqq Q^*[\lambda] \leqq \frac{1}{2}(pr)'/(pr) = \beta(t)$. Hence from Theorem 1, we find that $D(t_0) \leqq D(t) \leqq D(t_0)(pr/p_0r_0)^{\frac{1}{2}}$, where $p_0$ and $q_0$ are the values of $p$ and $r$ respectively when $t = t_0$. Here, of course, $D(t)$ represents the distance between two arbitrary solutions with respect to the metric appropriate to the value $t$ of the independent variable. The geodesics are straight lines, since the Riemann metric, though dependent upon $t$, is independent of $x_1$ and $x_2$. Thus, if we compare the distance between an arbitrary solution $(x_1(t), x_2(t))$ of (3.4) and the trivial solution $(0, 0)$, we see that (3.7) yields upper and lower bounds for the quantity $D(t) = (pr(x_1(t))^2 + (x_2(t))^2)^{\frac{1}{2}}$. This shows a fortiori that

$$(pr)^{\frac{1}{2}} \mid x_1(t) \mid \leqq D(t_0)(pr/p_0r_0)^{\frac{1}{2}}.$$

Hence, dividing by $(pr)^{\frac{1}{2}}$, we come out with the result of Wintner and Leighton, previously referred to, which states that every solution of (3.3) is bounded, if $q \equiv 0$, $p > 0$, $r > 0$, and $(pr)' \geqq 0$.

*Case* 2. $p \equiv 1$, $\lim_{t\to\infty} q(t) = 0$, $\lim_{t\to\infty} r(t) = R$, where $R$ is a positive constant. We also assume that

$$\int_{t_0}^{+\infty} \mid q'(t) \mid dt \text{ and } \int_{t_0}^{\infty} \mid r'(t) \mid dt$$

both converge for a suitably chosen value of $t_0$. From these special conditions, we easily see that the general condition, $pr - (1/4)(pq)^2 > 0$, is satisfied for $t \geqq t_0$, if $t_0$ is large enough; and, from (3.7) and (3.8), we see that both $\lambda^2$ and $\mid \lambda\mu \mid$ are bounded, say, $\lambda^2 \leqq A$, $\mid \lambda\mu \mid \leqq A$. Hence, from (3.6), we are justified in writing

$$\begin{aligned}\alpha(t) &= -\tfrac{1}{2}q(t) - \tfrac{1}{2}A(\mid q'(t) \mid + \mid r'(t) \mid) \\ &\leqq Q^*[\lambda] \leqq -\tfrac{1}{2}q(t) + \tfrac{1}{2}A(\mid q'(t) \mid + \mid r'(t) \mid) = \beta(t).\end{aligned}$$

Thus, from Theorem 1, we find that

$$D(t_0)K^{-1}\exp(-\tfrac{1}{2}\int_{t_0}^{t} q(u)\,du) \leqq D(t) \leqq D(t_0)K\exp(-\tfrac{1}{2}\int_{t_0}^{t} q(u)\,du),$$

where $K = \exp(+\tfrac{1}{2}A\int_{t_0}^{\infty}(\mid r'(t) \mid + \mid q'(t) \mid)dt)$.

Here $D(t)$, as in Case 1, may be taken to represent the distance between an arbitrary solution $(x_1(t), x_2(t))$ of (3.4) and the trivial solution $(0, 0)$, so that

$$D(t) = (rx_1^2 + qx_1x_2 + x_2^2)^{\frac{1}{2}} = [(r - \tfrac{1}{4}q^2)x_1^2 + (\tfrac{1}{2}qx_1 + x_2)^2]^{\frac{1}{2}}$$

It thus follows, for example, that

$$|x_1(t)| \leqq (r - \tfrac{1}{4}q^2)^{-\frac{1}{2}} KD(t_0) \exp(-\tfrac{1}{2}\int_{t_0}^{t} q(u)\,du)$$

$$\leqq K^* D(t_0) \exp(-\tfrac{1}{2}\int_{t_0}^{t} q(u)\,du),$$

where $K^*$ is a new constant determined in an obvious enough way.

Notice, for example, that the Bessel differential equation can be considered under both Case 1 and Case 2. Under Case 1, we write it in the form $(tx')' + (t - n^2t^{-1})x = 0$, taking $p = t$, $q = 0$, $r = t - n^2/t$, and we conclude that every solution is bounded for $t > t_0 > 0$. But, if we consider it under Case 2, by writing it in the form $x'' + t^{-1}x' + (1 - n^2t^{-2})x = 0$, taking $p \equiv 1$, $q = t^{-1}$, and $r = (1 - n^2t^{-2})$, we obtain the familiar sharper result that $x(t) = O(t^{-\frac{1}{2}})$.

**4. Approximation and linearization in connection with the present metric theory.** The effectiveness of any application of our theory depends upon the choice of the metric, that is of the functions $g_{ij}[x, t]$. Very little can be said in a completely general way as to the most efficient way of choosing this metric. In some important cases, however, the key to the situation can be found by a study of the linear case, as we explain more fully later on.

Suppose now that upon making certain approximations a system (2.2) reduces to a simpler system,

$$dx_i/dt = L^i[x, t], \qquad i = 1, \cdots, n. \tag{4.1}$$

Then, if the approximations are justified, it is clear that a metric suited for the study of (4.1) will also be suited for the study of (2.2). Moreover the validity of our approximations can be appraised with the help of this same metric, $g_{ij}dx_idx_j$. Namely we can introduce an extra variable $x_{n+1}$ and consider the $(n+1)$-*th* order system

$$\begin{aligned} &dx_i/dt = x_{n+1}L^i[x, t] + (1 - x_{n+1})X^i[x, t], \qquad i = 1, \cdots, n; \\ &dx_{n+1}/dt = 0 \end{aligned} \tag{4.2}$$

in connection with the $(n+1)$ dimensional metric, $\sum_{i,j=1}^{n} g_{ij}dx_idx_j + dx_{n+1}^2$.

Any solution of (4. 2) for which $x_{n+1} = 1$ yields a solution of (4. 1) and any solution of (4. 2) for which $x_{n+1} = 0$ yields a solution of (2. 2). Hence our Theorem 1 may be applied to compare those solutions of (2. 2) and (4. 1) which have the same initial values. These may be thought of as two solutions of the $(n+1)$-th order system (4. 2) which initially are at a unit's distance apart in the $(n+1)$ dimensional space. One solution lies on the integral manifold $x_{n+1} = 0$, the other on $x_{n+1} = 1$. The length of any arc joining a point on one of these manifolds to a point on the other manifold must be $\geqq 1$ and can be equal to 1 only if the $x_1, \cdots, x_n$ coordinates of both points are the same. This follows from the fact that $\sum_{i,j=1}^{n} g_{ij}x_ix_j$ is positive definite. Hence a rigorous measure of an upper bound for the deviation of the solution of the "approximate" equations (4. 1) from the solution of the "exact" equations (2. 2) could be taken to be $D(t) - 1$ or $(D(t)^2 - 1)^{\frac{1}{2}}$, where $D(t) \leqq \exp \int_{t_0}^{t} \beta(w)\,du$. Here, of course, $D(t)$ and $\beta$ are all formed with reference to the $(n+1)$ dimensional problem, and $\beta(t)$ is an upper bound of a certain quadratic form in the components of an $(n+1)$ dimensional unit vector (cf. (3. 2)).

Let us now consider equations (2. 2) together with approximations leading to equations (4. 1) which are supposed to split naturally into three groups. First, we have a set of $\mu$ equations involving only the unknowns, $x_1, \cdots, x_\mu$:

$$dx_i/dt = L^i(x_1, \cdots, x_\mu, t); \qquad i = 1, 2, \cdots, \mu. \tag{4. 3}$$

Secondly we have a set of $\nu$ equations which are linear in the $\nu$ unknowns $x_{\mu+1}, \cdots, x_{\mu+\nu}$, with coefficients which are functions of $x_1, \cdots, x_\mu$, and $t$:

$$\begin{aligned} dx_i/dt &= L^i(x_1, \cdots, x_{\mu+\nu}, t) \\ &= \sum_{j=\mu+1}^{\mu+\nu} a^i{}_j(x_1, \cdots, x_\mu, t)x_j + b^i(x_1, \cdots, x_\mu, t) \\ &\qquad i = \mu+1, \cdots, \mu+\nu. \end{aligned} \tag{4. 4}$$

Thirdly we have the remaining $\rho = n - (\mu + \nu)$ equations of completely general type:

$$dx_i/dt = L^i[x, t], \qquad i = \mu+\nu+1, \cdots, \mu+\nu+\rho = n. \tag{4. 5}$$

Here, we asume that $\rho \geqq 0$, and we need not exclude the vacuous case $\rho = 0$.

Such a situation arises in exterior ballistics, when for example we make a linearizing approximation by assuming known functions of time for the axial velocity, axial spin, and direction parameters for the shell axis. Denoting

these quantities by $x_1$, $x_2$, $x_3$, $x_4$, we can formulate these assumptions in the form of $\mu = 4$ differential equations of the form (4. 3) in which the right hand members depend simply on $t$. In such cases, of course, the integration of (4. 3) is immediately effected by quadratures. Upon substituting the resulting values of $x_1, \cdots, x_\mu$ in (4. 4), we obtain a linear system for $x_{\mu+1}, \cdots, x_{\mu+\nu}$ and it is here that the most interesting problems of stability arise in connection with the approximate ballistic equations, in which $\nu = 4$ and in which $x_5$, $x_6$, $x_7$, $x_8$ denote the cross components of velocity and spin. If first (4. 3) and then (4. 4) can be solved, the approximate solution is completed by solving (4. 5), an independent system of order $\rho$. In the ballistic case this can be done by quadratures. On account of the central importance of the linear system (4. 4) we shall therefore devote the following section to a discussion of a favorable metric for its study.

The metric so obtained will depend upon the coefficients of the linear system which in turn depend upon the particular solution of (4. 2), which has been inserted into these coefficients. In other words the metric of the linear system will be of the form,

$$\sum_{i,j=\mu+1}^{\mu+\nu} g_{ij}(x_1, \cdots, x_\mu, t)\, dx_i dx_j,$$

and then it is natural to set up the metric for the entire system (4. 1), and thus also for (2. 2) in the form,

$$\sum_{k=1}^{\mu} A_k dx_k^2 + \sum_{i,j=\mu+1}^{\mu+\nu} g_{ij}(x_1, \cdots, x_\mu, t)\, dx_i dx_j + \sum_{h=\mu+\nu+1}^{n} B_h dx_h^2,$$

where the $A$'s and $B$'s are merely positive constants chosen for convenience in weighting the importance of the different dependent variables. We cannot discuss here the choice of these constants.

**5. The metric for general linear systems.** In discussing the metric for the linear system (4. 4) we change our notation writing it in the form $dx_i/dt = \sum_{i,j=1}^{n} a_{ij}(t)x_j + b_i(t)$. In applications, of course, all derivatives of the $a$'s with respect to $t$ must be taken in the sense $\sum_{k=1}^{\mu} (\partial a/\partial x_k)L^k[x, t] + \partial a/\partial t$, if, as in (4. 4), the $a$'s depend on $t$ not only explicitly but through the mediation of $x_1, \cdots, x_\mu$.

THEOREM 3. *Given any non-singular real matrix $A$ with simple elemen-*

*tary divisors, it is always possible to find a positive definite symmetric matrix* $G$ *(of the same order as* $A$*) such that the roots* $\lambda_1, \cdots, \lambda_n$ *of the equation,*

$$\det((B - 2\lambda G) = 0, \text{ where } B = GA + A'G,$$

*are precisely the real parts of the characteristic roots of* $A$. If the elements of $A$ are functions of a variable $t$, it is of course necessary to assume that $A$ has simple elementary divisors for all values of $t$ for which $G = G(t)$ is required.

The following proof gives a construction of $G$:

*Proof.* If the complex characteristic roots of $A$ be denoted by $\alpha_k \pm i\beta_k$, $k = 1, 2, \cdots, s$, the transformation $y = Ax$ (where $x$ and $y$ represent vectors $x_1, \cdots, x_n$, and $y_1, \cdots, y_n$, respectively) can be written in the canonical form

$$\eta_{2k-1} = \alpha_k\xi_{2k-1} - \beta_k\xi_{2k}, \qquad \eta_{2k} = \beta_k\xi_{2k-1} + \alpha_k\xi_{2k}, \qquad k = 1, \cdots, s,$$

$$\eta_h = \gamma_h\xi_h, \qquad h = 2s + 1, \cdots, n,$$

when $\xi$ and $\eta$ are real vectors related to $x$ and $y$ by means of equations of the form, $\xi = Cx$, $\eta = Cy$, $\det C \neq 0$.
Let

$$Q = \sum_{j=1}^{n} \xi_j^2 = \sum_{j=1}^{n} \Big(\sum_{i=1}^{n} c_{ji}x_i\Big)^2 = \sum_{i,j} g_{ij}x_ix_j, \text{ where } g_{ij} = \sum_{h=1}^{n} c_{hi}c_{hj} = g_{ji}.$$

We regard $Q$ as an invariant with respect to transformations of coordinates. Hence the vector $\partial Q/\partial x_1, \cdots, \partial Q/\partial x_n$ is covariant, while the vector $y_1, \cdots, y_n$ can be regarded as contravariant (in spite of the employment of subscripts instead of the more usual superscripts for contravariant vectors). It follows from tensor analysis that $\bar{Q} = \sum_{i=1}^{n} y_i\partial Q/\partial x_i$ is also an invariant and hence that

$$\bar{Q} = \sum_{h=1}^{n} \eta_h \cdot \partial Q/\partial\xi_h = \sum_{k=1}^{s} 2\alpha_k(\xi^2_{2k-1} + \xi^2_{2k}) + \sum_{h=2s+1}^{n} 2\gamma\xi^2_h.$$

We see immediately, from this representation of the quadratic form $\bar{Q}$, that the stationary values of $\bar{Q}$ under the condition $Q = 1$ are precisely the real numbers $2\alpha_k, 2\gamma_h$. But these stationary values are also known to be the roots $\sigma_1, \cdots, \sigma_n$ of the equation $\det(B - \sigma G) = 0$ where $G = (g_{ij})$. The theorem follows at once when we replace $\sigma$ by $2\lambda$.

The significance of Theorem 3 becomes obvious when we consider a system of linear differential equations of the form, $dx_i/dt = \sum_j a_{ij}(t)x_j$. If $Q = \sum g_{ij}x_ix_j$ and $\dot{Q} = \sum \dot{g}_{ij}x_ix_j$ $(\dot{g}_{ij} = dg_{ij}/dt)$, we estimate the rate of growth of $Q$ as follows:

$$dQ/dt = 2\sum g_{ij}x_i dx_j/dt + \sum \dot{g}_{ij}x_i x_j = 2\sum_{i,j,k} g_{ij}a_{jk}x_i x_k + \sum_{i,j} \dot{g}_{ij}x_i x_j = \bar{Q} + \dot{Q},$$

where $\bar{Q}$, as above, is the quadratic form whose coefficients form the matrix $GA + A'G$. Hence

$$(dQ/dt)/Q = (\bar{Q} + \dot{Q})/Q \tag{5.1}$$

is less than or equal to the maximum value of $\bar{Q} + \dot{Q}$ under the condition $Q = 1$. This follows from the fact that the right member of (5.1) is homogeneous of degree 0 in $x_1, \cdots, x_n$. This is true no matter how $G$ is chosen, so long as it is positive definite. If, however, it is chosen in accordance with Theorem 3 and if the derivatives $\dot{a}_{ij}$ and hence $\dot{g}_{ij}$ are small (as in the ballistic case), we can say that the *maximum* value of $d \log Q/dt$ is not greater than twice the algebraically greatest real part of the characteristic roots of $A$ increased by a small quantity, namely the maximum of the form $\dot{Q}$ under the condition $Q = 1$. A similar statement may be made regarding the minimum value of $d \log Q/dt$.

It is easily verified, as one would expect, that, with our present $g_{ij}(t)$ taken as the coefficients of a Riemann metric depending on $t$ but independent of $x_1, \cdots, x_n$, and with $X'[x, t] = \sum_j a_{ij}(t)x_j + b_i(t)$, the $Q^*[\lambda]$ of Section 3 is here the same as one half the quadratic form $\bar{Q} + \dot{Q}$ with the $x$'s replaced by $\lambda$'s.

In the examples of Section 3, the matrix $G$, that is, the metric $g_{ij}dx_i dx_i$, was indeed chosen in accordance with Theorem 3. The two characteristic roots in each example were complex with real parts 0, in the first example (Case 1) and $\frac{1}{2}q$ in the second example (Case 2). Thus, it is seen in more detail that our fundamental Theorem 1 is capable of generalizing the rather refined results regarding these two examples to general linear systems in which the coefficient matrix has simple elementary divisors.

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# THE $m$-th POWER RESIDUE SYMBOL.*

By W. H. MILLS.

**1. Introduction.** Let $k$ be an algebraic number field containing the primitive $m$-th roots of unity. The $m$-th power residue symbol has been widely studied and many of its properties are known. The quotient $\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right)^{-1}$ has received much attention and has been evaluated in a number of special cases. In all these investigations the field $k$ has been held fixed. The object of this paper is to investigate the behavior of the power residue symbols when the field $k$ and the exponent $m$ are varied.

**2. Definitions.** Let $m$ be a positive integer and $k$ an algebraic number field containing a primitive $m$-th root of unity $\zeta$. Let $\mathfrak{p}$ be a prime ideal of $k$ not dividing $m$.

LEMMA 1. $\zeta^a \equiv \zeta^b \pmod{\mathfrak{p}}$ *if and only if* $\zeta^a = \zeta^b$.

*Proof.*

$$\prod_{r=1}^{m-1}(x-\zeta^r) = x^{m-1}+x^{m-2}+\cdots+1. \quad \text{If } x=1 \text{ then } \prod_{r=1}^{m-1}(1-\zeta^r) = m.$$

Suppose $\zeta^a \equiv \zeta^b \pmod{\mathfrak{p}}$. Then $\mathfrak{p} \mid 1-\zeta^{a-b}$. It follows that $\zeta^{a-b}=1$, and therefore $\zeta^a = \zeta^b$. The converse is immediate.

Let $\alpha$ be an element of $k$ relatively prime to $\mathfrak{p}$. Then $\alpha^{N\mathfrak{p}-1} \equiv 1 \pmod{\mathfrak{p}}$. In particular $\zeta^{N\mathfrak{p}-1} \equiv 1 \pmod{\mathfrak{p}}$, and so by Lemma 1, $\zeta^{N\mathfrak{p}-1}=1$. Therefore $m \mid N\mathfrak{p}-1$. Now $(\alpha^{(N\mathfrak{p}-1)/m})^m \equiv 1 \pmod{\mathfrak{p}}$. The equation $x^m \equiv 1 \pmod{\mathfrak{p}}$ has exactly $m$ solutions, namely $1, \zeta, \zeta^2, \cdots, \zeta^{m-1}$. Therefore $\alpha^{(N\mathfrak{p}-1)/m} \equiv \zeta^r \pmod{\mathfrak{p}}$ for some $r$.

*Definition* 1. $\left(\frac{\alpha}{\mathfrak{p}}\right)_m$ is the unique $m$-th root of unity satisfying

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_m \equiv \alpha^{(N\mathfrak{p}-1)/m} \pmod{\mathfrak{p}}.$$

*Definition* 2. Let A be a non-zero field element or ideal of $k$, relatively prime to $\alpha$ and $m$. Then

$$\left(\frac{\alpha}{\mathrm{A}}\right)_m = \prod_{\mathfrak{p}}\left(\frac{\alpha}{\mathfrak{p}}\right)_m^{\mathrm{ord}_{\mathfrak{p}}\mathrm{A}},$$

* Received January 25, 1950.

where the product is taken over all prime ideals of $k$ that are relatively prime to $\alpha$ and $m$.

The subscripts $k$ and $m$ will be dropped when it can be done without danger of ambiguity.

It is easily shown that $\left(\frac{\alpha}{\mathfrak{p}}\right) = 1$ if and only if $\alpha$ is congruent to an $m$-th power modulo $\mathfrak{p}$. For this reason the symbol $\left(\frac{\alpha}{\mathrm{A}}\right)$ is called an $m$-th power residue symbol. It is to be noted that it is defined if and only if $k$ is an algebraic number field containing the primitive $m$-th roots of unity, and A is a non-zero element or ideal of $k$ relatively prime to $m$ and $\alpha$. It will be assumed in the following discussion that all residue symbols occurring in the statements of lemmas and theorems are defined.

**3. Properties of the power residue symbol.** From Definitions 1 and 2 it follows immediately that $\left(\frac{\alpha}{\mathrm{A}}\right)$ is multiplicative in both $\alpha$ and A. Thus we have:

LEMMA 2. *The m-th power residue symbol is doubly multiplicative, i. e.*

$$\left(\frac{\alpha}{\mathrm{A}}\right)\cdot\left(\frac{\beta}{\mathrm{A}}\right)=\left(\frac{\alpha\beta}{\mathrm{A}}\right) \text{ and } \left(\frac{\alpha}{\mathrm{A}}\right)\cdot\left(\frac{\alpha}{\mathrm{B}}\right)=\left(\frac{\alpha}{\mathrm{AB}}\right).$$

We will now investigate how the value of the power residue symbol varies when the field $k$ and the power $m$ are varied with $\alpha$ and A held fixed. Because of the multiplicative properties of the residue symbol it will be sufficient to prove the following lemmas only for the case $\mathrm{A} = \mathfrak{p}$, a prime ideal of $k$. It follows from Lemma 1 that to prove two residue symbols equal, say an $m$-th power and an $n$-th power residue symbol, it is sufficient to prove them congruent modulo a prime ideal not dividing $mn$.

LEMMA 3. *Let K be a finite extension of k with degree n. Then*

$$\left(\frac{\alpha}{\mathrm{A}}\right)_{Km}=\left(\frac{\alpha}{\mathrm{A}}\right)^{n}_{km}.$$

*Proof.* Without loss of generality let $\mathrm{A} = \mathfrak{p}$, a prime ideal of $k$. Then $m \mid N\mathfrak{p} - 1$, so we may put $N\mathfrak{p} = 1 + bm$, $b$ a rational integer. Let $\mathfrak{P}$ be a prime ideal of $K$ such that $\mathfrak{P} \mid \mathfrak{p}$ with residue class field degree $f$.

$$(N\mathfrak{P}-1)/m = (N\mathfrak{p}^f-1)/m = ((1+bm)^f-1)/m \equiv bf = f(N\mathfrak{p}-1)/m \pmod{bm}.$$

Since $\alpha^{bm} = \alpha^{N\mathfrak{p}-1} \equiv 1 \pmod{\mathfrak{p}}$, it follows that

$$\left(\frac{\alpha}{\mathfrak{P}}\right)_{Km} \equiv \alpha^{(N\mathfrak{P}-1)/m} \equiv \alpha^{f(N\mathfrak{p}-1)/m} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)^{f}_{km} \pmod{\mathfrak{P}}.$$

Now $\mathfrak{P} \nmid m$. Therefore $\left(\frac{\alpha}{\mathfrak{P}}\right)_{Km} = \left(\frac{\alpha}{\mathfrak{p}}\right)_{km}^{f}$. Let $\mathfrak{p} = \prod \mathfrak{P}_\nu^{e_\nu}$ be the factorization of $\mathfrak{p}$ in $K$. Then

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{Km} = \prod \left(\frac{\alpha}{\mathfrak{P}_\nu}\right)_{Km}^{e_\nu} = \prod \left(\frac{\alpha}{\mathfrak{p}}\right)_{km}^{e_\nu f_\nu} = \left(\frac{\alpha}{\mathfrak{p}}\right)_{km}^{\Sigma e_\nu f_\nu} = \left(\frac{\alpha}{\mathfrak{p}}\right)_{km}^{n}.$$

LEMMA 4. *Let $\sigma$ be an automorphism of $k$. Then* $\sigma\left(\frac{\alpha}{\mathfrak{A}}\right) = \left(\frac{\sigma\alpha}{\sigma\mathfrak{A}}\right)$.

*Proof.* Without loss of generality let $\mathfrak{A} = \mathfrak{p}$. By definition

$$\alpha^{(N\mathfrak{p}-1)/m} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right) \pmod{\mathfrak{p}}, \text{ hence } (\sigma\alpha)^{(N\mathfrak{p}-1)/m} \equiv \sigma\left(\frac{\alpha}{\mathfrak{p}}\right) \pmod{\sigma\mathfrak{p}},$$

and since $N(\sigma\mathfrak{p}) = N\mathfrak{p}$ we have

$$\left(\frac{\sigma\alpha}{\sigma\mathfrak{p}}\right) \equiv (\sigma\alpha)^{(N\sigma\mathfrak{p}-1)/m} \equiv \sigma\left(\frac{\alpha}{\mathfrak{p}}\right) \pmod{\sigma\mathfrak{p}}.$$

Lemma 4 follows immediately.

LEMMA 5. *Let $\alpha \in k$, $\mathfrak{A}$ be a non-zero element or ideal of $k$, and let $K$ be a finite extension of $k$ containing primitive $n$-th roots of unity. Then* $\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kn} \in k$.

*Proof.* Put $L = k(\epsilon)$, where $\epsilon$ is a primitive $n$-th root of unity. Then $K \supseteq L$, and $L$ is a normal extension of $k$. Let $\sigma$ be any automorphism of $L/k$. Using Lemma 4 we obtain

$$\sigma\left(\frac{\alpha}{\mathfrak{A}}\right)_{Ln} = \left(\frac{\sigma\alpha}{\sigma\mathfrak{A}}\right)_{Ln} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{Ln}.$$

Therefore $\left(\frac{\alpha}{\mathfrak{A}}\right)_{Ln} \in k$. By Lemma 3 we have $\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kn} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{Ln}^{(K:L)} \in k$.

LEMMA 6. $\left(\frac{\alpha}{\mathfrak{A}}\right)_{k,mn}^{m} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{kn}$.

*Proof.* Without loss of generality let $\mathfrak{A} = \mathfrak{p}$. Then

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{k,mn}^{m} \equiv \alpha^{(N\mathfrak{p}-1)/n} \equiv \left(\frac{\alpha}{\mathfrak{p}}\right)_{kn} \pmod{\mathfrak{p}}.$$

Let $m = \prod l_\nu^{n_\nu}$ be the factorization of $m$ into powers of distinct rational primes. Let $a_\nu$ be rational integers satisfying $\sum a_\nu m l_\nu^{-n_\nu} = 1$. Then by Lemma 6

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{km} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{\Sigma a_\nu m l_\nu^{-n_\nu}} = \prod \left(\frac{\alpha}{\mathfrak{A}}\right)_{kl_\nu^{n_\nu}}^{a_\nu}.$$

Thus the study of the $m$-th power residue symbol can be reduced to the study of prime power residue symbols. Furthermore if $r < n$ then by Lemma

6 we can express the $l^r$-th power residue symbol in terms of the $l^n$-th power symbol. Therefore it is sufficient to consider the $l^n$-th power residue symbol, where $l$ is a prime and $k$ does not contain primitive $l^{n+1}$-st roots of unity. When $k$ contains the primitive $l^n$-th roots of unity but not the primitive $l^{n+1}$-st roots of unity we will say that $k$ contains the $l^n$-th roots of unity exactly.

Let $k$ contain the $l^n$-th roots of unity exactly, and let $K$ be a finite extension of $k$ containing $\zeta$, a primitive $l^{n+s}$ root of unity, $s \geqq 1$. We wish to express $\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kl^{n+s}}$ in terms of $\left(\frac{\alpha}{\mathfrak{A}}\right)_{kl^n}$. By Lemma 3 it is sufficient to consider the case $K = k(\zeta)$.

THEOREM 1. *Let $l$ be a prime and $n$ a non-negative integer. Let $k$ contain the $l^n$-th roots of unity exactly, and put $K = k(\zeta)$, where $\zeta$ is a primitive $l^{n+s}$ root of unity, $s \geqq 1$. Let $R$ be the field of rational numbers. Then*

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kl^{n+s}} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{kl^n}^{E},$$

*where* $E = \begin{cases} 1 & \text{if } l > 2, \\ 1 + 2^{n-1} & \text{if } l = 2,\ n > 1, \\ 2 & \text{if } l = 2,\ n = 1, \text{ and } k \cap R(\zeta) \text{ is a real field}, \\ 1 & \text{if } l = 2,\ n = 1, \text{ and } k \cap R(\zeta) \text{ is a complex field}. \end{cases}$

*Proof.* Case I. $(K : k) = l$. Let $G$ be the Galois group of $K/k$. $N_{K/k}\zeta = \prod_{\sigma \in G} \sigma\zeta = N_{R(\zeta)/k \cap R(\zeta)}\zeta$ will be evaluated.

(1) $l^n > 2$. Here $s = 1$ and

$$N_{K/k}\zeta = \prod_{\sigma \in G} \sigma\zeta = \prod_{\nu=0}^{l-1} \zeta^{1+\nu l^n} = \zeta^{l+l^{n+1}(l-1)/2} = \zeta^{l^s E}.$$

(2) $l^n = 1$. $N_{K/k}\zeta$ is an $l^s$ root of unity contained in $k$, but $k$ does not contain primitive $l$-th roots of unity. Therefore $N_{K/k}\zeta = 1 = \zeta^{l^s E}$. We note that the case $l = 2$, $n = 0$ is excluded by hypothesis.

(3) $l^n = 2$, $k \cap R(\zeta)$ is real. $N_{K/k}\zeta$ is a positive real number and a root of unity. Therefore $N_{K/k}\zeta = 1 = \zeta^{2l^s} = \zeta^{l^s E}$.

(4) $l^n = 2$, $k \cap R(\zeta)$ is complex. The field $R(\zeta)$ has three subfields of degree 2, the field of the $2^{n+s-1}$-st roots of unity which is fixed under the automorphism $\zeta \to -\zeta$, a real field fixed under $\zeta \to 1/\zeta$, and a complex field not containing $i$, namely $k \cap R(\zeta)$, which is fixed under the automorphism $\zeta \to -1/\zeta$. Thus $N_{K/k}\zeta = -1 = \zeta^{l^s E}$. Therefore $N_{K/k}\zeta = \zeta^{l^s E}$ in all cases. It follows immediately that if $\epsilon$ is any $l^{n+s}$ root of unity then $N_{K/k}\epsilon = \epsilon^{l^s E}$. Without loss of generality let $\mathfrak{A} = \mathfrak{p}$, a prime ideal of $k$. Let $\mathfrak{P}$ be a prime ideal of $K$ dividing $\mathfrak{p}$ with residue class field degree $f$. Then $f = 1$ or $f = l$.

(A) Suppose $f=1$. Then $\mathfrak{p}=\prod_{\sigma\in G}\sigma\mathfrak{P}$, and

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{Kl^{n+s}}=\prod_{\sigma\in G}\left(\frac{\alpha}{\sigma\mathfrak{P}}\right)_{Kl^{n+s}}=\prod_{\sigma\in G}\sigma\left(\frac{\alpha}{\mathfrak{P}}\right)_{Kl^{n+s}}=\left(\frac{\alpha}{\mathfrak{P}}\right)_{Kl^{n+s}}^{l^sE}$$
$$\equiv\alpha^{(N\mathfrak{P}-1)l^{-n}E}=\alpha^{(N\mathfrak{p}-1)l^{-n}E}\equiv\left(\frac{\alpha}{\mathfrak{p}}\right)_{kl^n}^{E}\pmod{\mathfrak{P}}.$$

(B) Suppose $f=l$. Then

$$(N\mathfrak{P}-1)/(N\mathfrak{p}-1)=(N\mathfrak{p}^l-1)/(N\mathfrak{p}-1)=1+N\mathfrak{p}+\cdots+N\mathfrak{p}^{l-1}.$$

Therefore by the theory of finite fields

$$\zeta^{(N\mathfrak{P}-1)/(N\mathfrak{p}-1)}\equiv\prod_{\sigma\in G}\sigma\zeta=\zeta^{l^sE}\qquad(\mathrm{mod}\,\mathfrak{p}).$$

It follows that $(N\mathfrak{P}-1)/(N\mathfrak{p}-1)\equiv l^sE\pmod{l^{n+s}}$ or

$$(N\mathfrak{P}-1)l^{-s}/(N\mathfrak{p}-1)\equiv E\qquad(\mathrm{mod}\,l^n).$$

Therefore

$$\left(\frac{\alpha}{\mathfrak{p}}\right)_{Kl^{n+s}}=\left(\frac{\alpha}{\mathfrak{P}}\right)_{Kl^{n+s}}\equiv\alpha^{(N\mathfrak{P}-1)l^{-n-s}}$$
$$\equiv\left(\frac{\alpha}{\mathfrak{p}}\right)_{kl^n}^{(N\mathfrak{P}-1)l^{-s}/(N\mathfrak{p}-1)}=\left(\frac{\alpha}{\mathfrak{p}}\right)_{kl^n}^{E}(\mathrm{mod}\,\mathfrak{P}).$$

This completes the proof of Theorem 1 for the case $(K:k)=l$. Now $(K:k)$ is a positive power of $l$. There remains:

Case II. $(K:k)=l^r$, $r\geqq 2$. We suppose that the theorem holds for $(K:k)=l^{r-1}$ and for $(K:k)=l$. Now $(k(\zeta^l):k)=l^{r-1}$ and $k(\zeta^l)$ contains the $l^{n+s-1}$-st roots of unity exactly. Thus we have

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kl^{n+s}}=\left(\frac{\alpha}{\mathfrak{A}}\right)_{k(\zeta^l),l^{n+s-1}}^{E'}=\left(\frac{\alpha}{\mathfrak{A}}\right)_{kl^n}^{E'E}$$

where $E'\equiv 1\pmod{l^{n+s-2}}$. Since $s\geqq r\geqq 2$ it follows that $E'E\equiv E\pmod{l^n}$. This completes the proof.

The result of Theorem 1 can be easily extended to arbitrary power residue symbols.

THEOREM 2. *Let $K$ be a finite extension of the algebraic number field $k$. Let $K$ contain the primitive $r$-th roots of unity. Let $r=\prod l_j^{s_j}$ be the factorization of $r$ into powers of distinct rational primes, and let $k$ contain the $l_j^{n_j}$ roots of unity exactly. Put $m=\prod l_j^{n_j}$. Let $\zeta_j$ be a primitive $l_j^{s_j}$-th root of unity. If $l_j^{n_j}>2$, or if $s_j\leqq n_j$, put $\Delta_j=0$; if $l_j=2$, $s_j>n_j=1$, and $\zeta_j\in k(i)$, put $\Delta_j=s_j-2$; and if $l_j=2$, $n_j=1$, and $\zeta_j\notin k(i)$, let $k(i)$ contain the $2^{2+\Delta_j}$-th roots of unity exactly. Put $k_j=k(\zeta_j)$ and*

$$E_j = \begin{cases} 1 & \text{if } s_j \leqq n_j, \\ 1 & \text{if } s_j > n_j,\ l_j > 2, \\ 1 + 2^{n-1} & \text{if } l_j = 2,\ s_j > n_j > 1, \\ 2 & \text{if } l_j = 2,\ s_j > n_j = 1, \text{ and } k \cap R(\zeta_j) \text{ is a real field}, \\ 1 & \text{if } l_j = 2,\ s_j > n_j = 1, \text{ and } k \cap R(\zeta_j) \text{ is a complex field}. \end{cases}$$

*Choose an integer $Q$ satisfing*

$$Qrl_j^{-s_j} \equiv E_j(K:k)\, m l_j^{\Delta_j - s_j} \pmod{l_j^{n_j}} \tag{1}$$

*for all $j$ for which $n_j > 0$. Then*

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kr} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{Q} \tag{2}$$

*for all $\alpha$ and $\mathfrak{A}$ for which both sides are defined.*

We note that the right side of the congruence (1) is integral if $n_j > 0$. Thus the integer $Q$ is determined modulo $m$.

*Proof.* (I) Suppose $s_j > n_j > 0$. Then $(k_j : k) = l_j^{s_j - n_j - \Delta_j}$. By Lemmas 3 and 6 and Theorem 1 we have

$$\begin{aligned} \left(\frac{\alpha}{\mathfrak{A}}\right)_{Kr}^{rl_j^{-s_j}} &= \left(\frac{\alpha}{\mathfrak{A}}\right)_{Kl_j^{s_j}} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{kj l_j^{s_j}}^{(K:kj)} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{kl_j^{n_j}}^{E_j(K:kj)} \\ &= \left(\frac{\alpha}{\mathfrak{A}}\right)_{kl_j^{n_j}}^{E_j(K:k)\, l_j^{n_j + \Delta_j - s_j}} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{Qrl_j^{-s_j}}. \end{aligned} \tag{3}$$

(II) Suppose $s_j \leqq n_j$. Then $\Delta_j = 0$ and $E_j = 1$. By Lemmas 3 and 6

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kr}^{rl_j^{-s_j}} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{kl_j^{s_j}}^{(K:k)} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{E_j(K:k)\, m l_j^{\Delta_j - s_j}} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{Qrl_j^{-s_j}} \tag{4}$$

(III) Suppose $n_j = 0$. Then $l_j \nmid m$ and $k$ does not contain primitive $l_j$-th roots of unity. Whence

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kr}^{mrl_j^{-s_j}} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{Kl_j^{s_j}}^{m} = 1 = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{Qmrl_j^{-s_j}}. \tag{5}$$

The exponents appearing on the left side of (3), (4), and (5) are relatively prime. (2) follows easily.

COROLLARY. Under the hypothesis of Theorem 2, if $r \mid (K:k)m$ then

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kr} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{(K:k)m/r} \text{ if } r \text{ is odd, and}$$

$$\left(\frac{\alpha}{\mathfrak{A}}\right)_{Kr} = \left(\frac{\alpha}{\mathfrak{A}}\right)_{km}^{(K:k)m(1+\frac{1}{2}m)/r} \text{ if } 4 \mid m.$$

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# RECIPROCITY IN ALGEBRAIC NUMBER FIELDS.*

By W. H. Mills.

**1. Introduction.** Let $m$ be a positive rational integer and $k$ be an algebraic number field containing primitive $m$-th roots of unity. Let $\alpha$ and $\beta$ be elements of $k$ that are relatively prime to each other and to $m$. The quotient of $m$-th power residue symbols, $\left(\frac{\alpha}{\beta}\right)\cdot\left(\frac{\beta}{\alpha}\right)^{-1}$, has been extensively studied [1] and many important properties of it are known. It is closely connected with the Hilbert norm residue symbol. Formulas for this quotient are known in many special cases, in particular for all cyclotomic fields and for certain extensions of cyclotomic fields.[2] The purpose of this paper is to obtain a general formula for this expression, valid for all algebraic number fields for which it is defined.[3]

**2. The norm residue symbol.** Let $R$ be the field of rational numbers, and $\bar{R}$ be the completion of $R$ at a prime $p$, not necessarily finite. Let $\bar{k}$ be a finite extension of $\bar{R}$ containing primitive $m$-th roots of unity. Let $\alpha$ and $\beta$ be non-zero elements of $\bar{k}$, and $\mathfrak{p}$ be the prime of $\bar{k}$.

*Definition.* The Hilbert norm residue symbol $(\alpha, \beta, \bar{k}, m)$ is that $m$-th root of unity satisfying [4]

$$(\alpha, \beta, \bar{k}, m)\beta^{1/m} = (\alpha, \bar{k}(\beta^{1/m})/\bar{k} \mid \mathfrak{p})\beta^{1/m}.$$

The norm residue symbol will be written $(\alpha, \beta)$ when it can be done without danger of ambiguity. It has been frequently written $\left(\frac{\alpha, \beta}{\mathfrak{p}}\right)$.

The following properties of the norm residue symbol are well known.[5]

I. $(\alpha\beta, \gamma) = (\alpha, \gamma)(\beta, \gamma)$.

* Received January 25, 1950.

[1] See, for example, [4], part II.

[2] Rothgiesser [6] has obtained formulas that are applicable in any finite extension of the field of the $m$-th roots of unity that is unramified at all primes dividing $m$.

[3] In some recent papers, [7] and [8], Šafarevič has approached this problem using the $p^n$-th primary numbers introduced by Hasse [5]. His results are of a different nature from those of this paper.

[4] For details see [4], part II, p. 53.

[5] See [4], part II, pp. 54-59 and p. 75.

II. $(\alpha, \beta\gamma) = (\alpha, \beta)(\alpha, \gamma)$.

III. $(1 - \alpha, \alpha) = 1$.

IV. $(\beta, \alpha) = (\alpha, \beta)^{-1}$.

V. $(\alpha, \beta, k, mn)^m = (\alpha, \beta, k, n)$.

VI. Let $\alpha$ and $\beta$ be non-zero elements of an algebraic number field $k$ such that, if $\mathrm{ord}_\mathfrak{q} \alpha \neq 0$ and $\mathrm{ord}_\mathfrak{q} \beta \neq 0$, then $\mathfrak{q} \mid m$. Let $p_\infty$ be the infinite prime of the rational field. Put $\mathfrak{a} = \prod_{\mathfrak{p} \nmid m p_\infty} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p} \alpha}$ and $\mathfrak{b} = \prod_{\mathfrak{p} \nmid m p_\infty} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p} \beta}$. Then

$$\left(\frac{\beta}{\mathfrak{a}}\right)_{km} \cdot \left(\frac{\alpha}{\mathfrak{b}}\right)_{km}^{-1} = \prod_{\mathfrak{p} \mid m p_\infty} (\alpha, \beta, k_\mathfrak{p}, m),$$

where $k_\mathfrak{p}$ is the $\mathfrak{p}$-adic completion of $k$.

VII. If $m > 1$ then

$$\prod_{\mathfrak{p} \mid p_\infty} (\alpha, \beta, k_\mathfrak{p}, m) = (-1)^{N(k, \alpha, \beta)},$$

where $N(k, \alpha, \beta)$ is the number of isomorphisms $\sigma$ of $k$ into the field of real numbers such that $\sigma\alpha < 0$, $\sigma\beta < 0$. If $m > 2$ then $N(k, \alpha, \beta) = 0$ since $k$ contains primitive $m$-th roots of unity and must therefore be a complex field.

An immediate consequence of VI and VII is:

VIII. If $\alpha$, $\beta$, and $m$ are relatively prime in pairs and if $m > 1$, then

$$\left(\frac{\beta}{\alpha}\right)_{km} \cdot \left(\frac{\alpha}{\beta}\right)_{km}^{-1} = (-1)^{N(k, \alpha, \beta)} \prod_{\mathfrak{p} \mid m} (\alpha, \beta, k_\mathfrak{p}, m).$$

IX. If $\bar{K}$ is a finite extension of $\bar{k}$, $\beta \in \bar{k}$, then

$$(\alpha, \beta, \bar{K}, m) = (N_{\bar{K}/\bar{k}}\alpha, \beta, \bar{k}, m).$$

IX follows from the known result $\left(\frac{K\Omega/K}{\mathfrak{a}}\right) = \left(\frac{\Omega/k}{N_{\bar{K}/\bar{k}}\mathfrak{a}}\right)$. A special case of IX is:

X. If $\bar{k}$ does not contain i, $\bar{K} = \bar{k}(i)$, and if $\alpha$, $\beta$, and $\gamma$ are non-zero elements of $\bar{k}$, then $(\alpha^2 + \beta^2, \gamma, \bar{k}, 2) = (\alpha - \beta i, \gamma, \bar{K}, 2)$.

The problem of evaluating the reciprocity quotient is reduced by VIII to the problem of evaluating the norm residue symbol $(\alpha, \beta, k_\mathfrak{p}, m)$, where $\mathfrak{p} \mid m$. It is sufficient to consider the case $m = l^n$, $l$ prime, $n \geqq 1$. For this case we need the following theorem:

THEOREM 1. *Let $\bar{R}$ be the completion of the rational field $R$ at a finite prime $l$. Let $\bar{k}$ be a finite extension of $\bar{R}$ containing the $l^n$-th roots of unity*

*exactly,*[6] *where* $l^n > 2$. *Put* $\bar{K} = \bar{k}(\zeta)$, $\zeta$ *a primitive* $l^{n+s}$ *root of unity,* $s \geqq 1$. *Then*

$$(\alpha, \beta, \bar{k}, l^n) = (\alpha, \beta, \bar{K}, l^{n+s})^E,$$

*where* $E = 1$ *if* $l > 2$, *and* $E = 1 + 2^{n-1}$ *if* $l = 2$.

*Proof.* Let $\mathfrak{p}$ and $\mathfrak{P}$ be the prime ideals of $\bar{k}$ and $\bar{K}$ respectively. Let $K'$ be an algebraic number field with a prime $\mathfrak{P}'$ such that $\bar{K}$ is the $\mathfrak{P}'$-adic completion of $K'$. Put $K = K'(\zeta)$. Then $K$ has a prime $\mathfrak{P}$ such that $\bar{K}$ is the $\mathfrak{P}$-adic completion of $K$. Put $k = K \cap \bar{k}$. Then $k$ contains the $l^n$-th roots of unity exactly. Furthermore $k$ contains a prime ideal $\mathfrak{p}$ such that $\bar{k}$ is the $\mathfrak{p}$-adic completion of $k$, $K/k$ is isomorphic to $\bar{K}/\bar{k}$, and $K = k(\zeta)$. By the approximation theorem $\alpha'$ and $\beta'$ can be chosen in $k$ such that:

(1) $\alpha'/\alpha$ and $\beta'/\beta$ are $l^{n+s}$ powers in $\bar{k}$ and therefore in $\bar{K}$,

(2) if $\mathfrak{q} \neq \mathfrak{p}$, then $\operatorname{ord}_\mathfrak{q} \alpha' \neq 0$ or $\operatorname{ord}_\mathfrak{q} \beta' \neq 0$, and

(3) $\alpha'$ is an $l^{n+s}$ power in $k_\mathfrak{q}$ for all primes $\mathfrak{q} \neq \mathfrak{p}$, $\mathfrak{q} \mid l p_\infty$.

From (1) it follows that $(\alpha, \beta, \bar{k}, l^n) = (\alpha', \beta', \bar{k}, l^n) = (\alpha', \beta', k_\mathfrak{p}, l^n)$. From (3) it follows that $(\alpha', \beta', k_\mathfrak{q}, l^n) = 1$ if $\mathfrak{q} \neq \mathfrak{p}$, $\mathfrak{q} \mid l p_\infty$. Put $\mathfrak{a} = \prod_{\mathfrak{q} \nmid l p_\infty} \mathfrak{q}^{\operatorname{ord}_\mathfrak{q} \alpha'}$ and $\mathfrak{b} = \prod_{\mathfrak{q} \nmid l p_\infty} \mathfrak{q}^{\operatorname{ord}_\mathfrak{q} \beta'}$. Then by VI we have

$$(\alpha, \beta, \bar{k}, l^n) = \prod_{\mathfrak{q} \mid l p_\infty} (\alpha', \beta', k_\mathfrak{q}, l^n) = \left(\frac{\beta'}{\mathfrak{a}}\right)_{k l^n} \cdot \left(\frac{\alpha'}{\mathfrak{b}}\right)_{k l^n}^{-1}.$$

Similarly

$$(\alpha, \beta, \bar{K}, l^{n+s}) = \left(\frac{\beta'}{\mathfrak{a}}\right)_{K l^{n+s}} \cdot \left(\frac{\alpha'}{\mathfrak{b}}\right)_{K l^{n+s}}^{-1}.$$

In the preceding paper it was proved that if $k$ is an algebraic number field containing the $l^n$-th roots of unity exactly and $K = k(\zeta)$, then $\left(\frac{\gamma}{\mathfrak{A}}\right)_{K l^{n+s}} = \left(\frac{\gamma}{\mathfrak{A}}\right)_{k l^n}^E$, where $E = 1$ if $l > 2$, and $E = 1 + 2^{n-1}$ if $l = 2$, $n > 1$.

Applying this result we obtain $(\alpha, \beta, \bar{K}, l^{n+s}) = (\alpha, \beta, \bar{k}, l^n)^E$. Now $E^2 \equiv 1 \pmod{l^n}$. Therefore $(\alpha, \beta, \bar{k}, l^n) = (\alpha, \beta, \bar{k}, l^n)^{E^2} = (\alpha, \beta, \bar{K}, l^{n+s})^E$.

Simple expressions for $(\alpha, \beta)$ are known in a number of special cases.[7] In particular if $\omega$ is a root of unity of order prime to $m$ then clearly

$$(\alpha, \omega, \bar{k}, m) = 1. \tag{4}$$

If $\epsilon$ is a root of unity whose order is a power of the prime $l$, $\mathfrak{p}$ the prime of $\bar{k}$, $\mathfrak{p} \mid l$, $l^n > 2$, and if $\alpha \equiv 1 \pmod{\mathfrak{p}}$ is an element of $\bar{k}$, then

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[6] When a field $k$ contains primitive $l^n$-th roots of unity but not primitive $l^{n+1}$-st roots of unity, $k$ is said to contain the $l^n$-th roots of unity exactly. Cf. the preceding paper.

[7] See [4], part II, pp. 74-113, and [1].

$$(5) \qquad (\alpha, \epsilon, \bar{k}, l^n) = \epsilon^{El^{-n}S_{\bar{k}/\bar{R}}\log\alpha},$$

where $\bar{R}$ is the $l$-adic completion of the rational field, $E=1$ if $l>2$, $E=1+2^{n-1}$ if $l=2$, and $\log\alpha$ is the $\mathfrak{p}$-adic logarithm defined by

$$\log\alpha = -\sum_{\nu=1}^{\infty}(1-\alpha)^\nu/\nu.$$

If $\bar{L}=\bar{R}(\zeta)$, $\zeta$ a primitive $l^n$-th root of unity, $\Theta=1/(\zeta-1)$, and $1-\alpha$ belongs to the prime ideal of $\bar{L}$, then

$$(6) \qquad (\alpha, 1-\zeta, \bar{L}, l^n) = \zeta^{l^{-n}S_{\bar{L}/\bar{R}}(\zeta\Theta\log\alpha)}.$$

A weak consequence of (5) and (6) is:

LEMMA 1. *Let $\bar{L}=\bar{R}(\zeta)$, where $\zeta$ is a primitive $l^n$-th root of unity. Let $\bar{k}$ be a finite extension of $\bar{L}$ with the prime ideal $\mathfrak{p}$. Then if $\gamma\in\mathfrak{p}$,*

$$l^n \mid S_{\bar{k}/\bar{R}}\{\sum_{\nu=0}^{\infty}\gamma^{l^\nu}l^{-\nu}/(\zeta-1)\}.$$

*Proof.* Using well known properties of the $\mathfrak{p}$-adic logarithm [8] we obtain

$$\begin{aligned} S_{\bar{k}/\bar{R}}\{\sum_{\nu=0}^{\infty}\gamma^{l^\nu}l^{-\nu}/(\zeta-1)\} &= S_{\bar{L}/\bar{R}}(\Theta S_{\bar{k}/\bar{L}}\sum_{\nu=0}^{\infty}\gamma^{l^\nu}l^{-\nu}) \\ &= -S_{\bar{L}/\bar{R}}(\Theta S_{\bar{k}/\bar{L}}\log\beta) = S_{\bar{L}/\bar{R}}((1-\zeta\Theta)\log N_{\bar{k}/\bar{L}}\beta) \\ &= S_{\bar{L}/\bar{R}}\log\delta - S_{\bar{L}/\bar{R}}(\zeta\Theta\log\delta), \end{aligned}$$

for suitable $\beta\in\bar{k}$, $\delta\in\bar{L}$. The terms of this last expression are divisible by $l^n$ by (5) and (6) respectively.

**3. Computation of $(1-\omega\pi^{al^r}, \pi)$.** Let $l^n$ be any prime power greater than 2, $\bar{R}$ the completion of the rational field at the prime $l$, and $\bar{k}$ any finite extension of $\bar{R}$ containing the $l^n$-th roots of unity exactly. Let $\pi$ be any local uniformizing parameter of $\bar{k}$, and $\omega$ any root of unity contained in $\bar{k}$, whose order $m$ is relatively prime to $l$. Let $a$ be a positive integer not divisible by $l$. The expression $(1-\omega\pi^{al^r}, \pi, \bar{k}, l^n)$ will be computed.

Let $\omega_0$ be the $m$-th root of unity satisfying $\omega_0{}^{l^r}=\omega$. We write $\omega_0=\omega^{l^{-r}}$. Let $\zeta$ be a primitive $l^n$-th root of unity. If $n\geqq r$ put $\epsilon=\zeta^{l^{n-r}}$, and if $n<r$ let $\epsilon$ be a primitive $l^r$-th root of unity satisfying $\zeta=\epsilon^{l^{r-n}}$. In either case $\epsilon^{\xi l^{r-n}}$ shall mean $\zeta^\xi$, for integral $\xi$. Put $\bar{K}=\bar{k}(\epsilon)$ and $n'=\mathrm{Max}(n,r)$. Then we have $(\bar{K}:\bar{k})=l^{n'-n}$. We will write $S$ and $S'$ for $S_{\bar{k}/\bar{R}}$ and $S_{\bar{K}/\bar{R}}$ respectively.

Put $E'=\begin{cases}1+2^{n-1} & \text{if } l=2, r>n,\\ 1 & \text{otherwise,}\end{cases}$ and $E''=\begin{cases}1+2^{n'-1} & \text{if } l=2,\\ 1 & \text{if } l>2.\end{cases}$

[8] See [5].

Then by Theorem 1, properties I, II, and III of the norm residue symbol, and (4) and (5) we have

$$(1-\omega\pi^{al^r},\pi,\bar{k},l^n)^a=(1-\omega\pi^{al^r},\pi,\bar{K},l^{n'})^{aE'}=\prod_{\nu=0}^{l^r-1}(1-\epsilon^\nu\omega_0\pi^a,\pi^a,\bar{K},l^{n'})^{E'}$$

$$=\prod_{\nu=0}^{l^r-1}(1-\epsilon^\nu\omega_0\pi^a,\epsilon^{-\nu}\omega_0^{-1},\bar{K},l^{n'})^{E'}=\prod_{\nu=0}^{l^r-1}(1-\epsilon^\nu\omega_0\pi^a,\epsilon,\bar{K},l^{n'})^{-\nu E'}$$

$$=\prod_{\nu=0}^{l^r-1}\epsilon^{-\nu E'E''l^{-n'}S'\log(1-\epsilon^\nu\omega_0\pi^a)}.$$

Now if $l>2$ then $E'E''=1=E$, and if $l=2$ then $E'E''\equiv 1+2^{n-1}=E \pmod{l^n}$. Thus in either case $E'E''\equiv E \pmod{l^n}$. Now $(1-\omega\pi^{al^r},\pi,\bar{k},l^n)$ is an integral power of $\zeta$, say $\zeta^A$. Then modulo $l^n$ we have

$$aA\equiv -El^{n-r-n'}\sum_{\nu=0}^{l^r-1}\nu S'\log(1-\epsilon^\nu\omega_0\pi^a)$$

$$=El^{n-r-n'}S'\{\sum_{\nu=0}^{l^r-1}\sum_{\mu=1}^{\infty}\nu\epsilon^{\nu\mu}\omega_0^\mu\pi^{a\mu}/\mu\}=ES\{\sum_{\mu=1}^{\infty}\omega_0^\mu\pi^{a\mu}F_1(\mu)/\mu\}$$

where $F_1(\mu)=l^{n-n'-r}S_{\bar{K}/\bar{k}}\{\sum_{\nu=0}^{l^r-1}\nu\epsilon^{\nu\mu}\}$. If $\epsilon^{\nu\mu}\varepsilon\bar{k}$ then $S_{\bar{K}/\bar{k}}\epsilon^{\nu\mu}=(\bar{K}:\bar{k})\epsilon^{\nu\mu}=l^{n'-n}\epsilon^{\nu\mu}$, and if $\epsilon^{\nu\mu}\notin\bar{k}$ then $S_{\bar{K}/\bar{k}}\epsilon^{\nu\mu}=0$. Therefore $F_1(\mu)=l^{-r}\sum\nu\epsilon^{\nu\mu}$, where the summation is over those $\nu$ for which $\epsilon^{\nu\mu}\varepsilon\bar{k}$, $0\leqq\nu<l^r$.

Now $\sum_{\xi=0}^{l^s-1}x^\xi=(x^{l^s}-1)/(x-1)$. Differentiating and multiplying by $x$ we obtain

$$\sum_{\xi=0}^{l^s-1}\xi x^\xi=l^sx^{l^s}/(x-1)-x(x^{l^s}-1)/(x-1)^2.$$

Therefore if $\lambda^{l^s}=1$, $\lambda\neq 1$, then

$$\sum_{\xi=0}^{l^s-1}\xi\lambda^\xi=l^s/(\lambda-1). \tag{7}$$

i) Suppose $l^r\mid\mu$. Then $\epsilon^{\nu\mu}=1\varepsilon\bar{k}$ for all $\nu$, and

$$F_1(\mu)=l^{-r}\sum_{\nu=0}^{l^r-1}\nu=\tfrac{1}{2}(l^r-1).$$

ii) Suppose $l^r\nmid\mu$, $l^{n'-n}\mid\mu$. Then $\epsilon^{\nu\mu}\varepsilon\bar{k}$ for all $\nu$, and by (7)

$$F_1(\mu)=l^{-r}\sum_{\nu=0}^{l^r-1}\nu\epsilon^{\nu\mu}=1/(\epsilon^\mu-1).$$

iii) Suppose $\mu = l^{n'-n-b}\mu_0$, $b > 0$, $l \nmid \mu_0$. Then $\epsilon^{\nu\mu} \in \bar{k}$ if and only if $l^b \mid \nu$. Making the substitution $\nu = l^b\xi$, and using (7)

$$F_1(\mu) = l^{-r} \sum \nu\epsilon^{\nu\mu} = l^{-r} \sum_{\xi=0}^{l^{r-b}-1} l^b\xi\epsilon^{l^b\xi\mu} = 1/(\epsilon^{\mu l^b} - 1).$$

$$\text{Summarizing, } F_1(\mu) = \begin{cases} \frac{1}{2}(l^r - 1) & \text{if } l^r \mid \mu, \\ 1/\epsilon^\mu - 1) & \text{if } l^r \nmid \mu,\ l^{n'-n} \mid \mu, \\ 1/\epsilon^{\mu l^b} - 1) & \text{if } l^{n'-n-b} \parallel \mu,\ b > 0. \end{cases}$$

Define $F_2(\mu) = -\frac{1}{2}$ if $l^r \mid \mu$, and $F_2(\mu) = F_1(\mu)$ if $l^r \nmid \mu$. Then

$$\sum_{\mu=1}^{\infty} \omega_0^{\mu}\pi^{a\mu}[F_1(\mu) - F_2(\mu)]/\mu = \sum_{l^r \mid \mu} \tfrac{1}{2}\omega_0^{\mu}\pi^{a\mu}l^r/\mu = -\tfrac{1}{2}\log(1 - \omega_0^{l^r}\pi^{al^r})$$
$$= -\tfrac{1}{2}\log(1 - \omega\pi^{al^r}).$$

Therefore $A \equiv ES\{\sum_{\mu=1}^{\infty} \omega_0^{\mu}\pi^{a\mu}F_2(\mu)/(a\mu)\} - \frac{1}{2}(E/a)\, S\log(1 - \omega\pi^{al^r}) \pmod{l^n}$.

Now by (5), if $\alpha \equiv 1 \pmod{\pi}$, then $l^n \mid ES\log\alpha$. Since $l \nmid E$, $l \nmid a$, it follows that $-\frac{1}{2}(E/a)S\log\alpha \equiv \frac{1}{2}S\log\alpha \pmod{l^n}$ for all $\alpha \equiv 1 \pmod{\pi}$. Therefore $A \equiv ES\{\sum_{\mu=1}^{\infty} \omega_0^{\mu}\pi^{a\mu}F_2(\mu)/(a\mu)\} + \frac{1}{2}S\log(1 - \omega\pi^{al^r}) \pmod{l^n}$. Furthermore if $l > 2$ the second term can be dropped. Now we put $\Delta(\mu) = 1/(\zeta^c - 1)$, where $\mu = l^\nu c$, $l \nmid c$. Then by Lemma 1

$$S\{\sum_{\mu=1}^{\infty} \omega_0^{\mu}\pi^{a\mu}\Delta(\mu)/(a\mu)\} = \sum_{l \mid c} (ac)^{-1} S\{\sum_{p=0}^{\infty} (\omega_0^{c}\pi^{ac})^{l^p} l^{-p}/(\zeta^c - 1)\} \equiv 0 \pmod{l^n}.$$

We note that if $l^{n'-n} \nmid \mu$ then $F_2(\mu) = \Delta(\mu)$. We now adopt the convention that $S(\gamma\pi^t) = 0$ if $\gamma$ is a non-zero element of $\bar{k}$ and $t$ is not an integer. Then making the substitution $\mu = \xi l^{r-n}$ we obtain, modulo $l^n$,

$$ES\{\sum_{\mu=1}^{\infty} \omega_0^{\mu}\pi^{a\mu}F_2(\mu)/(a\mu)\} \equiv ES\{\sum_{\mu=1}^{\infty} \omega_0^{\mu}\pi^{a\mu}[F_2(\mu) - \Delta(\mu)]/(a\mu)\}$$
$$= ES\{\sum_{l^{n'-n} \mid \mu} \omega_0^{\mu}\pi^{a\mu}[F_2(\mu) - \Delta(\mu)]/(a\mu)\}$$
$$= ES\{\sum_{\xi=1}^{\infty} \omega_0^{\xi l^{r-n}}\pi^{a\xi l^{r-n}}[F_2(\xi l^{r-n}) - \Delta(\xi l^{r-n})]/(a\xi l^{r-n})\}$$
$$= ES\{\sum_{\xi=1}^{\infty} \omega^{\xi l^{r-n}}\pi^{a\xi l^{r-n}}F_\zeta(\xi)/(a\xi l^{r-n})\},$$

where $F_\zeta(\xi) = F_2(\xi l^{r-n}) - \Delta(\xi l^{r-n})$. If $\xi = cl^\nu$, $l \nmid c$, then

$$F_\zeta(\xi) = \begin{cases} 1/(\zeta^\xi - 1) - 1/(\zeta^c - 1) & \text{if } l^n \nmid \xi. \\ -\frac{1}{2} - 1/(\zeta^c - 1) & \text{if } l^n \mid \xi, \end{cases}$$

Combining results we have

$$(8)\quad \begin{aligned} &(1-\omega\pi^{al^r},\pi,\bar{k},l^n)=\zeta^A,\\ &A=ES\{\sum_{\xi=1}^{\infty}\omega^{\xi l^{r-n}}\pi^{a\xi l^{r-n}}F_\xi(\xi)/(a\xi l^{r-n})+\tfrac{1}{2}S\log(1-\omega\pi^{al^r})\}. \end{aligned}$$

In deriving (8) the assumption was made that $l^n>2$. A similar result will now be derived for the case $l^n=2$.

Let $\bar{R}$ be the completion of the rational field at the prime 2. Let $\bar{k}$ be a finite extension of $\bar{R}$ not containing $i$. Let $\pi$ be a local uniformizing parameter of $\bar{k}$, and $\omega$ be a root of unity of odd order. We shall begin by computing $(1+\omega\pi^{a2^r},\pi,\bar{k},2)$, where $a$ is odd and $r\geqq 1$. Put $\bar{L}=\bar{k}(i)$. Let $\bar{L}$ contain the $2^{2+\delta}$ roots of unity exactly, and $\rho$ be a primitive $2^{2+\delta}$ root of unity satisfying $\rho^{2^\delta}=i$. If $r\leqq\delta+1$ put $\epsilon=\rho^{2^{\delta+1-r}}$, and if $r>\delta+1$ let $\epsilon$ be a primitive $2^{r+1}$-st root of unity satisfying $\epsilon^{2^{r-1-\delta}}=\rho$. In either case $\epsilon^{\xi 2^{r-1-\delta}}$ shall mean $\rho^\xi$, for integral $\xi$. Put $\bar{K}=\bar{L}(\epsilon)$. We will write $S$, $S'$, and $\bar{S}$ for $S_{\bar{k}/\bar{R}}$, $S_{\bar{K}/\bar{R}}$, and $S_{\bar{L}/\bar{k}}$ respectively. Put $\omega_0=\omega^{2^r}$, that is $\omega_0$ is the root of unity of odd order satisfying $\omega_0{}^{2^r}=\omega$. Put $n'=\mathrm{Max}(r+1,2+\delta)$. Then by properties $V$ and $X$ the norm residue symbol and Theorem 1 we have

$$\begin{aligned} (1+\omega\pi^{a2^r},\pi,\bar{k},2)&=(1-i\omega_0{}^{2^{r-1}}\pi^{a2^{r-1}},\pi,\bar{L},2)\\ &=(1-i\omega_0{}^{2^{r-1}}\pi^{a2^{r-1}},\pi,\bar{L},2^{2+\delta})^{2^{1+\delta}}=(1-i\omega_0{}^{2^{r-1}}\pi^{a2^{r-1}},\pi,\bar{K},2^{n'})^{2^{1+\delta}E'}, \end{aligned}$$

where $E'\equiv 1\pmod 2$. Since $(1+\omega\pi^{a2^r},\pi,\bar{k},2)$ is a square root of unity $E'$ may be replaced by 1. Proceeding as in the case $l^n>2$,

$$\begin{aligned} (1+\omega\pi^{a2^r},\pi,\bar{k},2)&=(1-i\omega_0{}^{2^{r-1}}\pi^{a2^{r-1}},\pi,\bar{K},2^{n'})^{2^{1+\delta}}\\ &=\prod_{\nu=0}^{2^{r-1}-1}(1-\epsilon^{4\nu+1}\omega_0\pi^a,\pi,\bar{K},2^{n'})^{2^{1+\delta}}\\ &=\prod_{\nu=0}^{2^{r-1}-1}(1-\epsilon^{4\nu+1}\omega_0\pi^a,\epsilon,\bar{K},2^{n'})^{-2^{1+\delta}(4\nu+1)/a}=(-1)^A, \end{aligned}$$

$$\begin{aligned} aA&\equiv -(1+2^{n'-1})2^{-r-n'+1+\delta}\sum_{\nu=0}^{2^{r-1}-1}(4\nu+1)S'\log(1-\epsilon^{4\nu+1}\omega_0\pi^a)\\ &\equiv 2^{-r-n'+1+\delta}\sum_{\nu=0}^{2^{r-1}-1}(4\nu+1)S'\{\sum_{\mu=1}^{\infty}\epsilon^{(4\nu+1)\mu}\omega_0{}^\mu\pi^{a\mu}/\mu\}\\ &=S\{\sum_{\mu=1}^{\infty}\omega_0{}^\mu\pi^{a\mu}F_1(\mu)/\mu\}\qquad(\mathrm{mod}\,2), \end{aligned}$$

where $F_1(\mu)=2^{-r-n'+1+\delta}\bar{S}\{\sum_{\nu=0}^{2^{r-1}-1}(4\nu+1)S_{\bar{K}/\bar{L}}\epsilon^{(4\nu+1)\mu}\}$.

i) Suppose $2^{r-1} \mid \mu$. Then $\epsilon^\mu \varepsilon \bar{L}$ and $\epsilon^{4\nu\mu} = 1$ for all $\nu$. Therefore $S_{\bar{K}/\bar{L}}\epsilon^{(4\nu+1)\mu} = (\bar{K}:\bar{L})\epsilon^\mu = 2^{n'-2-\delta}\epsilon^\mu$. Thus in this case

$$F_1(\mu) = 2^{-r-1}\sum_{\nu=0}^{2^{r-1}-1}(4\nu+1)\bar{S}\epsilon^\mu = \tfrac{1}{4}(2^r-1)\bar{S}\epsilon^\mu$$

$$= \begin{cases} \frac{1}{2}(-1)^{\mu 2^{-r}}(2^r-1) & \text{if } 2^r \mid \mu, \\ 0 & \text{if } 2^{r-1} \parallel \mu. \end{cases}$$

ii) Suppose $2^{r-1} \nmid \mu$, $2^{r-1-\delta} \mid \mu$. Then $\epsilon^{(4\nu+1)\mu} \varepsilon \bar{L}$, $\epsilon^{4\mu} \neq 1$. Using (7) we have

$$F_1(\mu) = 2^{-r-1}\bar{S}(\epsilon^\mu \sum_{\nu=0}^{2^{r-1}-1}(4\nu+1)\epsilon^{4\nu\mu})$$

$$= 2^{-r-1}\bar{S}(\epsilon^\mu \sum_{\nu=0}^{2^{r-1}-1} 4\nu\epsilon^{4\nu\mu}) = \bar{S}(\epsilon^\mu/(\epsilon^{4\mu}-1)).$$

iii) Suppose $2^{r-1-\delta} \nmid \mu$. Then $\epsilon^\mu \notin \bar{L}$ and so $\epsilon^{(4\nu+1)\mu} \notin \bar{L}$ which implies that $\bar{S}\epsilon^{(4\nu+1)\mu} = 0$. Therefore $F_1(\mu) = 0$. Summarizing

$$(9) \qquad F_1(\mu) = \begin{cases} \frac{1}{2}(-1)^{\mu 2^{-r}}(2^r-1 & \text{if } 2^r \mid \mu, \\ \bar{S}(\epsilon^\mu/(\epsilon^{4\mu}-1)) & \text{if } 2^{r-1} \nmid \mu,\ 2^{r-1-\delta} \mid \mu, \\ 0 & \text{otherwise.} \end{cases}$$

We now make the assertion that for $2 \nmid a$, $r \geqq 0$,

$$(10) \qquad (1+\omega(-\pi)^{a2^r}, \pi, \bar{k}, 2) = (-1)^A,$$

where $A \equiv S\{\sum_{\mu=1}^{\infty}\omega_0^{\mu}\pi^{a\mu}F_1(\mu)/(a\mu)\} \pmod 2$, $F_1(\mu)$ defined by (9). If $r \geqq 1$, then $(-\pi)^{a2^r} = \pi^{a2^r}$ and (10) follows immediately. If $r = 0$, then

$$(1+\omega(-\pi)^{a2^r}, \pi, \bar{k}, 2) = (1-\omega\pi^a, \pi, \bar{k}, 2) = (1-\omega\pi^a, \omega, \bar{k}, 2)^{-1/a} = 1.$$

In this case $2^r = 1$, and hence $F_1(\mu) = 0$ for all $\mu$. Thus $A$ is even and the proof of (10) is complete. Put $F_2(\mu) = -\frac{1}{2}(-1)^{\mu 2^{-r}}$ if $2^r \mid \mu$, and $F_2(\mu) = F_1(\mu)$ if $2^r \nmid \mu$. Then

$$S\{\sum_{\mu=1}^{\infty}\omega_0^{\mu}\pi^{a\mu}[F_1(\mu)-F_2(\mu)]/(a\mu)\} = S\{\sum_{2^r \mid \mu}\omega_0^{\mu}\pi^{a\mu}(-1)^{\mu 2^{-r}}2^{r-1}/(a\mu)\}$$

$$\equiv S\{\sum_{\xi=1}^{\infty}\tfrac{1}{2}(-1)^{\xi}\omega_0^{\xi 2^r}\pi^{a\xi 2^r}/\xi\} = -\tfrac{1}{2}S\log(1+\omega_0^{2^r}\pi^{a2^r})$$

$$= -\tfrac{1}{2}\log N_{\bar{k}/\bar{R}}(1+\omega_0^{2^r}\pi^{a2^r}) \equiv 0 \pmod 2,$$

since 4 divides the 2-adic logarithm of any odd integer in $\bar{R}$. As before we

adopt the convention that $S(\gamma\pi^t) = 0$ if $\gamma$ is a non-zero element of $\bar{k}$ and $t$ is not an integer. Then we have, modulo 2,

$$A \equiv S\{\sum_{\mu=1}^{\infty} \omega_0^{\mu}\pi^{a\mu}F_2(\mu)/(a\mu)\}$$

$$= S\{\sum_{\xi=1}^{\infty} \omega_0^{\xi 2^{r-1-\delta}}\pi^{a\xi 2^{r-1-\delta}}F_2(\xi 2^{r-1-\delta})/(a\xi 2^{r-1-\delta})\}$$

$$= S\{\sum_{\xi=1}^{\infty} \omega^{\xi 2^{r-1-\delta}}\pi^{a\xi 2^{r-1-\delta}}F_{-1}(\xi)/(a\xi 2^{r-1-\delta})\},$$

where

$$F_{-1}(\xi) = \begin{cases} -\frac{1}{2}(-1)^{\xi 2^{r-1-\delta}} & \text{if } 2^{1+\delta} \mid \xi, \\ 0 & \text{if } 2^{\delta} \parallel \xi, \\ \bar{S}(\epsilon^{\xi 2^{r-1-\delta}}/(\epsilon^{\xi 2^{r-1-\delta}} = 1)) & \text{if } 2^{\delta} \nmid \xi. \end{cases}$$

The results of this section will now be combined into one theorem.

THEOREM 2. *Let $l$ be a rational prime and $n$ a positive integer. Let $\bar{R}$ be the completion of the rational field at $l$, and let $\bar{k}$ be a finite extension of $\bar{R}$ containing the $l^n$-th roots of unity exactly. Let $\pi$ be any local uniformizing parameter of $\bar{k}$, and $\omega$ any root of unity of $\bar{k}$, whose order $m$ is relatively prime to $l$. Let $\zeta$ be a primitive $l^n$-th root of unity. Let the field obtained from $\bar{k}$ by adjoining the primitive $l^{n+1}$-st roots of unity contain the $l^{n+1+\delta}$ roots of unity exactly. If $l^n = 2$ let $\rho$ be a primitive $2^{n+1+\delta}$ root of unity. If $l^n > 2$, then $\delta = 0$. Put*

$$\eta = \begin{cases} -1 & \text{if } l^n = 2, \\ +1 & \text{if } l^n > 2, \end{cases} \quad \text{and} \quad E = \begin{cases} 1 + 2^{n-1} & \text{if } l = 2, n > 1, \\ 1 & \text{otherwise.} \end{cases}$$

*Let $S$ stand for $S_{\bar{k}/\bar{R}}$, with the convention noted above, and let $\omega^{\xi l^{-t}}$ be defined as the $m$-th root of unity $\omega'$ satisfying $\omega'^{l^t} = \omega^{\xi}$. Put*

$$F_{\zeta}(\xi) = \begin{cases} -\frac{1}{2} - 1/(\zeta^c - 1) & \text{if } l^n > 2, \xi = cl^v, l \nmid c, v \geqq n, \\ 1/(\zeta^{\xi} - 1) - 1/(\zeta^c - 1) & \text{if } l^n > 2, \xi = cl^v, l \nmid c, v < n, \\ -\frac{1}{2}(-1)^{\xi 2^{-1-\delta}} & \text{if } l^n = 2, 2^{1+\delta} \mid \xi, \\ 0 & \text{if } l^n = 2, 2^{\delta} \parallel \xi, \\ S_{\bar{k}(i)/\bar{k}}(\rho^{\xi}/(\rho^{4\xi} - 1)) & \text{if } l^n = 2, 2^{\delta} \nmid \xi. \end{cases}$$

*Then, if $l \nmid a$,*

$$(1 - \eta\omega(\eta\pi)^{al^r}, \pi, \bar{k}, l^n) = \zeta^A,$$

*where*

$$A = ES\{\sum_{\xi=1}^{\infty} \omega^{\xi l^{-n-\delta}}\pi^{a\xi l^{r-n-\delta}}F_{\zeta}(\xi)/(a\xi l^{r-n-\delta})\} + \tfrac{1}{4}(\eta + 1)S\log(1 - \eta\omega(\eta\pi)^{al^r}).$$

**4. A formula for $(\alpha, \pi)$.** We will now use Theorem 2 to derive a

formula for $(\alpha, \pi, \bar{k}, l^n)$, valid when $\alpha \equiv 1 \pmod{\pi}$. We need the following lemma:

LEMMA 2. *Let $\mu(N)$ be the Moebius $\mu$ function, and*

$$\mu^*(M) = \begin{cases} \mu(M) & \text{if } l^n > 2, \\ -\mu(M) & \text{if } l^n = 2, M \text{ odd}, \\ -2^{r-1}\mu(N) & \text{if } l^n = 2, M = 2^r N, N \text{ odd}, r \geqq 1. \end{cases}$$

*Then*

$$\sum_{d|M} \eta^{M/d}\mu^*(d) = \begin{cases} 1 \text{ if } M = 1, \\ 0 \text{ if } M > 1, \end{cases}$$

*where $\eta$ is as defined in Theorem 2.*

*Proof.* Case I. $l^n > 2$. In this case $\eta = 1$, $\mu^*(d) = \mu(d)$ and therefore

$$\sum_{d|M} \eta^{M/d}\mu^*(d) = \sum_{d|M} \mu(d) = \begin{cases} 1 \text{ if } M = 1, \\ 0 \text{ if } M > 1. \end{cases}$$

Case II. $l^n = 2$. In this case $\eta = -1$. Put $M = 2^r N$, $N$ odd. Making the substitution $d = 2^\nu D$, $D$ odd, we obtain

$$\begin{aligned} \sum_{d|M} \eta^{M/d}\mu^*(d) &= \sum_{\nu=0}^{r} \sum_{D|N} (-1)^{2^{r-\nu}} \mu^*(2^\nu D) \\ &= -\left((-1)^{2^r} + \sum_{\nu=1}^{r} (-1)^{2^{r-\nu}} 2^{\nu-1}\right) \sum_{D|N} \mu(D). \end{aligned}$$

Now

$$(-1)^{2^r} + \sum_{\nu=1}^{r} (-1)^{2^{r-\nu}} 2^{\nu-1} = \begin{cases} -1 \text{ if } r = 0, \\ 0 \text{ if } r > 0, \end{cases}$$

and

$$\sum_{D|N} \mu(D) = \begin{cases} 1 \text{ if } N = 1, \\ 0 \text{ if } N > 1. \end{cases}$$

Lemma 2 follows immediately.

A necesary and sufficient condition for the series $\sum_{\nu=1}^{\infty} a_\nu$ to converge in $k$ is that $\operatorname{ord}_\mathfrak{p} a_\nu \to \infty$ as $\nu \to \infty$, where $\mathfrak{p}$ is the prime of $\bar{k}$. Furthermore any convergent series in $\bar{k}$ is absolutely convergent. It follows that we can manipulate with much greater freedom than in ordinary analysis. In particular

$$E\sum_{\xi=1}^{\infty}\omega^{\xi l^{-n-\delta}}\pi^{a\xi l^{r-n-\delta}}F_\zeta(\xi)/(a\xi l^{r-n-\delta})$$

$$=E\sum_{\xi=1}^{\infty}F_\zeta(\xi)\sum_{\phi=1}^{\infty}(\omega^{\phi\xi l^{-n-\delta}}\pi^{a\phi\xi l^{r-n-\delta}}/(\phi^2 a\xi l^{r-n-\delta}))\sum_{\nu|\phi}\mu^*(\nu)\eta^{\phi/\nu}$$

$$=E\sum_{\xi=1}^{\infty}F_\zeta(\xi)\sum_{\nu=1}^{\infty}(\mu^*(\nu)/\nu)\sum_{\theta=1}^{\infty}\eta^\theta\omega^{\theta\nu\xi l^{-n-\delta}}\pi^{a\theta\nu\xi l^{r-n-\delta}}/(\theta^2\nu a\xi l^{r-n-\delta}) \tag{11}$$

$$=E\sum_{\tau=1}^{\infty}\sum_{\nu|\tau}(F_\zeta(\tau/\nu)\mu^*(\nu)/\nu)\sum_{\theta=1}^{\infty}\eta^\theta\omega^{\theta\tau l^{-n-\delta}}\pi^{a\theta\tau l^{r-n-\delta}}/a\theta^2\tau l^{r-n-\delta})$$

$$=\sum_{\tau=1}^{\infty}J_\zeta(\tau)\sum_{\theta=1}^{\infty}\eta^\theta\omega^{\theta\tau l^{-n-\delta}}\pi^{a\theta\tau l^{r-n-\delta}}/(a\theta^2\tau l^{r-n-\delta}),$$

where $J_\zeta(\tau)=E\sum_{\nu|\tau}F_\zeta(\tau/\nu)\mu^*(\nu)/\nu$.

Let $f$ be the residue class field degree of the extension $\bar{k}/\bar{R}$. Let $w$ be a primitive $l^f-1$ root of unity. Let $\alpha\equiv 1 \pmod{\pi}$ be an element of $\bar{k}$. Let $G(x,y)=1+\sum_{\nu=1}^{\infty}\sum_{\mu=0}^{l^f-2}a_{\nu\mu}x^\mu y^\nu$, where the $a_{\nu\mu}$ are rational integers such that $G(w,\eta\pi)=\alpha$. Then $\log G(x,y)=-\sum_{\nu=1}^{\infty}[1-G(x,y)]^\nu/\nu$ is a power series in $x$ and $y$ with rational coefficients, say

$$\log G(x,y)=\sum_{\nu=1}^{\infty}\sum_{\mu=0}^{\nu(l^f-2)}b_{\nu\mu}x^\mu y^\nu.$$

*Definition.* $[\alpha,\tau l^{-t}\mid G,\pi]=-\sum_{v=1}^{\infty}\sum_{\mu=0}^{v(l^f-2)}b_{\nu\mu}w^{\mu\tau l^{-t}}\pi^{\nu\tau l^{-t}}/(\nu\tau l^{-t})$.

It is easily seen that the series for $[\alpha,\tau l^{-t}\mid G,\pi]$ converges and that $[\alpha_1,\tau l^{-t}\mid G_1,\pi]+[\alpha_2,\tau l^{-t}\mid G_2,\pi]=[\alpha_1\alpha_2,\tau l^{-t}\mid G_1G_2,\pi]$.

THEOREM 3. *Let* $G(x,y)=1+\sum_{\nu=1}^{\infty}\sum_{\mu=0}^{l^f-2}a_{\nu\mu}x^\mu y^\nu$, $a_{\nu\mu}$ *rational integers such that* $G(w,\eta\pi)=\alpha$. *Then*

$$(\alpha,\pi,\bar{k};l^n)=\zeta^B, \tag{12}$$

*where*

$$B=S\{\sum_{\tau=1}^{\infty}J_\zeta(\tau)[\alpha,\tau l^{-n-\delta}\mid G,\pi]\}+\tfrac{1}{4}(\eta+1)S\log\alpha.$$

*Proof.* $G(x,y)$ can be written as an infinite product

$$G(x,y)=\prod_{v=1}^{\infty}\prod_{u=0}^{v(l^f-2)}\{G_{vu}(x,y)^{c_{vu}},$$

where $G_{vu}(x, y) = 1 - \eta x^u y^v$, and the $c_{vu}$ are rational integers. Since both sides of (12) are multiplicative in $\alpha$, it is sufficient to prove (12) for the case $G(x, y) = 1 - \eta x^u y^v$. In this case the desired result follows immediately from (11) and Theorem 2.

**5. The reciprocity quotient.** Let $m$ be a rational integer greater than 1, and let $k$ be an algebraic number field containing primitive $m$-th roots of unity. Let $m = \prod l_\nu^{n_\nu}$ be the factorization of $m$ into rational primes. Let $a_\nu$ be rational integers such that $\sum a_\nu m l_\nu^{-n_\nu} = 1$. Let $N(k, \alpha, \beta)$ be the number of isomorphisms $\sigma$ of $k$ into the field of real numbers such that $\sigma\alpha < 0$, $\sigma\beta < 0$. For each prime ideal $\mathfrak{p}$ dividing $m$ let $k_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of $k$. If $\mathfrak{p}$ divides the rational prime $l$, let $k_\mathfrak{p}$ contain the $l^{n\mathfrak{p}}$-th roots of unity exactly, and let $\zeta_\mathfrak{p}$ be a primitive $l^{n\mathfrak{p}}$-th root of unity. If $\alpha$ and $\beta$ are elements of $k$ that are relatively prime to each other and to $m$, we have

$$\left(\frac{\alpha}{\beta}\right)_{km} \cdot \left(\frac{\beta}{\alpha}\right)_{km}^{-1} = \prod \left\{ \left(\frac{\alpha}{\beta}\right)_{kl_\nu^{n_\nu}} \cdot \left(\frac{\beta}{\alpha}\right)_{kl_\nu^{n_\nu}}^{-1} \right\}^{a_\nu}$$
$$= (-1)^{N(k,\alpha,\beta)} \prod_\nu \prod_{\mathfrak{p}|l_\nu} (\alpha, \beta, k_\mathfrak{p}, l_\nu^{n_\nu})^{-a_\nu}.$$

Let $\pi$ be an element of $k$ such that $\pi \equiv \mathfrak{p}$, $\pi \not\equiv \mathfrak{p}^2$ for all prime ideals $\mathfrak{p}$ dividing $m$. Then for all such $\mathfrak{p}$, $\pi$ and $\beta\pi$ are local uniformizing parameters for $k_\mathfrak{p}$, and

$$(\alpha, \beta, k_\mathfrak{p}, l_\nu^{n_\nu}) = (\alpha, \beta\pi, k_\mathfrak{p}, l_\nu^{n\mathfrak{p}})^{l_\nu^{n\mathfrak{p}-n_\nu}} \cdot (\alpha, \pi, k_\mathfrak{p}, l_\nu^{n\mathfrak{p}})^{-l_\nu^{n\mathfrak{p}-n_\nu}}.$$

Now $\alpha^{N\mathfrak{p}-1} \equiv 1 \pmod{\mathfrak{p}}$, and if power series $G_\mathfrak{p}(x, y)$ and $H_\mathfrak{p}(x, y)$ are suitably chosen we can apply Theorem 3 and obtain:

THEOREM 4.

$$\left(\frac{\alpha}{\beta}\right)_{km} \cdot \left(\frac{\beta}{\alpha}\right)_{km}^{-1} = (-1)^{N(k,\alpha,\beta)} \prod_\nu \prod_{\mathfrak{p}|l_\nu} \zeta_\mathfrak{p}^{C\mathfrak{p}a_\nu l_\nu^{n\mathfrak{p}-n_\nu}/(N\mathfrak{p}-1)},$$

where

$$C_\mathfrak{p} = S_{k_\mathfrak{p}/R_{l_\nu}} \{ \sum_{\tau=1}^{\infty} J_{\zeta_\mathfrak{p}}(\tau) ([\alpha^{N\mathfrak{p}-1}, \tau l_\nu^{-n-\delta} \mid G_\mathfrak{p}, \pi] - [\alpha^{N\mathfrak{p}-1}, \tau l_\nu^{-n-\delta} \mid H_\mathfrak{p}, \beta\pi]) \}.$$

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# EXTREMAL PROBLEMS IN THE THEORY OF BOUNDED ANALYTIC FUNCTIONS.*

By ZEEV NEHARI.

**1. Introduction and notations.** Let $B$ be the family of analytic functions $f(z)$ which are regular and single-valued and satisfy $|f(z)| \leq 1$ in a multiply-connected domain $D$. Generalizing the classical case in which $D$ is the unit sircle, it is possible to pose a large number of extremal problems for continuous functionals defined on $B$. Some problems of this kind were solved by H. Grunsky [5, 6], L. V. Ahlfors [1], P. R. Garabedian [4] and the author [8]. In the present paper, a method is developed which makes it possible to solve a large variety of such problems; the method works equally well when the class $B$ is restricted by various additional conditions. In all cases, the extremal functions yield $(1, m)$ conformal maps of $D$ onto the unit circle, where $m$ depends on the particular extremal problem and can be explicitly stated in each case. The method consists in associating with each given problem an allied extremal problem within a family of specified functions involving a finite number of parameters. The boundary values of the solution of the associated problem are used to set up a Dirichlet problem, which is then shown to yield the solution of the original problem.

We shall use the following notations:

$D$ denotes a finite domain in the complex $z$-plane bounded by $n$ closed analytic curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$; the boundary of $D$ will be denoted by $\Gamma$, that is, $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$. The assumption that the boundary components are analytic curves is only made for the convenience of the proofs; by standard arguments, all results can be readily extended to the case of general boundary components.

$B$ denotes the family of analytic functions $f(z)$ which are regular, single-valued, and bounded, say $|f(z)| \leq 1$, in $D$.

$B_R$ denotes the family of analytic functions $f(z)$ which are regular and single-valued and have a bounded real part, say $|\operatorname{Re}\{f(z)\}| \leq 1$, in $D$. The classes $B$ and $B_R$ can be transformed into each other by an elementary conformal mapping, and extremal problems in one class are equivalent to corresponding extremal problems in the other. Since, from the point of view

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of the method of this paper, the class $B_R$ is much easier to handle than the class $B$, most proofs will be carried out for functions of $B_R$.

$g(z,\zeta)$ denotes the Green's function of $D$ with respect to the point $\zeta \in D$, i. e. $g(z,\zeta) + \log|z-\zeta|$ is harmonic in $D$ and $g(z,\zeta)$ vanishes for $z \in \Gamma$.

$N(z,\zeta)$ denotes the Neumann function of $D$ with respect to the point $\zeta \in D$, i. e. $N(z,\zeta) + \log|z-\zeta|$ is harmonic in $D$ and the normal derivative of $N(z,\zeta)$ is constant on $\Gamma$.

$\omega_\nu(z)$ denotes the harmonic measure of $\Gamma_\nu$ with respect to $D$, i. e. $\omega_\nu(z)$ is harmonic in $D$, and it has the boundary values 1 and 0 on $\Gamma_\nu$ and $\Gamma - \Gamma_\nu$, respectively.

**2. The Schwarz lemma.** In order to illustrate the method of proof in a comparatively simple case, we first consider the generalization of the classical Schwarz lemma, that is, the problem: $f(z) \in B$, $|f'(\zeta)| = \max.$ $(\zeta \in D)$ which was treated—by different methods—in [1, 4, 8]. We note that the maximizing function $F(z)$ must vanish at $z = \zeta$, since otherwise the function [1]

$$F_1(z) = \frac{F(z) - F(\zeta)}{1 - [F(\zeta)]^* F(z)} = \frac{F'(\zeta)}{1 - |F(\zeta)|^2}(z-\zeta) + O(|z-\zeta|^2)$$

would satisfy $|F'_1(\zeta)| > |F'(\zeta)|$. We may also assume that $F'(\zeta) > 0$ since, for $|\gamma| = 1$, $\gamma F(z)$ is also in $B$.

The transformation $w_1 = (4/\pi)\tan^{-1} w$ maps the circle $|w| < 1$ conformally onto the infinite strip $-1 < \mathrm{Re}\{w_1\} < 1$. Hence, the function

$$\phi(z) = (4/\pi)\tan^{-1} f(z) \tag{1}$$

is in $B_R$. Since $F(\zeta) = 0$ and $F'(\zeta) > 0$, we may confine ourselves to functions $f(z)$ which likewise satisfy these conditions. We then have

$$\phi(\zeta) = 0, \qquad \phi'(\zeta) = (4/\pi) f'(\zeta) > 0. \tag{2}$$

If $\zeta = \xi + i\eta$, the above problem is therefore equivalent to the following problem concerning the class $B_R$: $\phi(z) \in B_R$, $\phi(\zeta) = 0$, $\mathrm{Re}\{\phi'(\zeta)\} = \max.$ We note that in carrying through the proof we may confine ourselves to functions $\phi(z)$ which are continuous in $D + \Gamma$. Once the result is obtained for this restricted class, the general case then follows by a standard argument involving the approximation of $D$ by a sequence of domains interior to it.

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[1] Complex conjugates are denoted by asterisks.

By Green's formula, we have

$$(3)\qquad \mathrm{Re}\{\phi(\zeta)\} = u(\zeta) = -\tfrac{1}{2\pi}\int_\Gamma u(z)\,(\partial g(z,\zeta)/\partial n)\,ds, \qquad ds = |\,dz\,|,$$

where $\phi(z) = u(z) + iv(z)$, and $\partial/\partial n$ denotes differentiation with respect to the outwards pointing normal. Differentiating (3) with respect to $\xi$, we obtain

$$(4)\qquad \phi'(\zeta) = \mathrm{Re}\{\partial\phi(\zeta)/\partial\xi\} = \partial w(\zeta)/\partial\xi = -\tfrac{1}{2\pi}\int_\Gamma u(z)\,(\partial^2 g(z,\zeta)/\partial n\partial\xi)\,ds.$$

We also have

$$(5)\qquad \int_\Gamma u(z)\,(\partial\omega_\nu(z)/\partial n)\,ds = 0, \qquad \nu = 1, \cdots, n.$$

Indeed by Green's formula, the Cauchy-Riemann equations, and the boundary properties of $\omega_\nu(z)$,

$$(5')\qquad \int_\Gamma u(\partial\omega_\nu/\partial n)\,ds = \int_\Gamma \omega_\nu(\partial u/\partial n)\,ds = \int_{\Gamma_\nu}(\partial u/\partial n)\,ds$$

$$= \int_{\Gamma_\nu}(\partial v/\partial s)\,ds = \int_{\Gamma_\nu} dv,$$

and this vanishes, since the function $\phi(z) = u(z) + iv(z)$ is single-valued in $D$. We further note that

$$(6)\qquad \int_\Gamma u(z)\,(\partial g(z,\zeta)/\partial n)\,ds = 0,$$

since $u(\zeta) = 0$. It follows from (4), (5), and (6) that

$$(7)\qquad u_\xi(\zeta) = \phi'(\zeta)$$

$$= -\tfrac{1}{2\pi}\int_\Gamma u(z)\,[\partial^2 g(z,\zeta)/\partial n\partial\xi + \alpha(\partial g(z,\zeta)/\partial n) + \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu(z)/\partial n)]\,ds,$$

where $\alpha, \lambda_1, \cdots, \lambda_{n-1}$ are arbitrary real constants.

Since $|\,u(z)\,| \leq 1$ in $D$, we have

$$(8)\qquad |\,u_\xi(\zeta)\,|$$

$$\leq \tfrac{1}{2\pi}\int_\Gamma |\,\partial^2 g(z,\zeta)/\partial n\partial\xi + \alpha(\partial g(z,\zeta)/\partial n) + \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu(z)/dn)\,|\,ds.$$

For any choice of the arbitrary parameters $\alpha$, $\lambda_\nu$, (8) yields an upper bound for $|\,u_\xi(\zeta)\,|$. In order to obtain the best possible upper bound of this type, we minimize the right-hand side of (8) with respect to the parameters

$\alpha, \lambda_1, \cdots, \lambda_{n-1}$. This is a problem in ordinary calculus, which certainly has at least one solution. Indeed, as shown by (8), the right-hand side of (8) is bounded from below; moreover, any converging subsequence of a minimizing sequence (in the variables $\alpha, \lambda_\nu$) must converge to a finite limit. If this were not the case and one or more of the variables tended to $\infty$ or $-\infty$, there would exist one variable, say $\lambda_\mu$, such that $\limsup |\lambda_\nu/\lambda_\mu| \leq 1$ $(\nu \neq \mu)$, $\limsup |\alpha/\lambda_\mu| \leq 1$. Since we are dealing with a minimizing sequence, the right-hand side of (8) will be bounded; if we divide the integrand by $|\lambda_\mu|$, the value of the integral will tend to zero. This, however, involves a contradiction, since we may choose a subsequence for which the integrand tends to a well-defined positive function; it cannot vanish identically, since the coefficient of $\partial\omega_\mu/\partial n$ is 1.

Let now $\alpha, \lambda_1, \cdots, \lambda_{n-1}$ denote the particular values of these parameters which minimize the right-hand side of (8). With the abbreviation

$$P = \partial^2 g(z,\zeta)/\partial n \partial\xi + \alpha(\partial g(z,\zeta)/\partial n) + \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu(z)/\partial n),$$

it follows from the minimum property that

$$\int_\Gamma |P + \epsilon(\partial\omega_\nu/\partial n)|\, ds \geq \int_\Gamma |P|\, ds, \tag{9}$$

where $\epsilon$ is a small real parameter. Similarly,

$$\int_\Gamma |P + \epsilon(\partial g(z,\zeta)/\partial n)|\, ds \geq \int_\Gamma |P|\, ds. \tag{10}$$

(9) is equivalent to $\epsilon \int_\Gamma |P|/P(\partial\omega_\nu/\partial n)\, ds + o(\epsilon) \geq 0$, whence, in view of the arbitrariness of the sign of $\epsilon$,

$$\int_\Gamma |P|/P(\partial\omega_\nu/\partial n)\, ds = 0. \tag{11}$$

In the same way, (10) leads to

$$\int_\Gamma |P|/P(\partial g(z,\zeta)/\partial n)\, ds = 0. \tag{12}$$

The function $|P|/P$ is not defined at points at which $P = 0$. This, however, does not affect the integrals (11) and (12) since these points are finite in number (it is easy to show that the number of zeros of $P$ on $\Gamma$ cannot exceed $2n$) and we may there assign to $P$ arbitrary values and understand the integrals in the Lebesgue sense. Another possibility is that of breaking

$\Gamma$ up into arcs connecting two adjacent zeros of $P$ and extending the definition of $|P|/P$ to the ends of these arcs by continuity. Evidently, both procedures give identical results. Whenever integrals of the type (11) and (12) will appear in the sequel, it will always be assumed, without being expressly stated each time, that the definition of the integrand has been completed by one or the other of these procedures.

We now introduce the bounded function $U(z)$ which is harmonic in $D$ and takes the boundary values $-1$ or $+1$ at points at which $P$ is positive or negative, respectively. We have $U(z) = -|P|/P$, $z \in \Gamma$, $P \neq 0$. Although the boundary values at the finite number of points at which $P = 0$ are not specified, $U(z)$ is uniquely determined by this Dirichlet problem in view of the additional assumption that $U(z)$ be bounded. $U(z)$ is easily seen to be of the form

$$U(z) = \frac{1}{2\pi} \int_\Gamma |P|/P (dg(z,t)/\partial n_t)\, ds_t.$$

In terms of this function $U(z)$, conditions (11) and (12) can be rewritten in the form

$$(11') \qquad \int_\Gamma U(z)\,(\partial \omega_\nu(z)/\partial n)\, ds = 0, \qquad \nu = 1, \cdots, n-1,$$

$$(12') \qquad \int_\Gamma U(z)\,(\partial g(z,\zeta)/\partial n)\, ds = 0.$$

(12′) expresses the fact that $U(\zeta) = 0$. If $V(z)$ denotes the harmonic conjugate of $U(z)$, it follows from (11′) and a manipulation similar to (5′) that $V(z)$ is free of periods about the boundary components $\Gamma_\nu$, $\nu = 1, \cdots, n-1$ (and therefore also about $\Gamma_n$). $V(z)$ is only determined up to an arbitrary constant, which will be so chosen that $V(\zeta) = 0$.

The analytic function $\Phi(z) = U(z) + iV(z)$ is thus found to have the following properties: $\Phi(z)$ is regular and single-valued in $D$; on $\Gamma$, we have, except at a finite number of points, $|\mathrm{Re}\{\Phi(z)\}| = 1$. Since $|\mathrm{Re}\{\Phi(z)\}| \leq 1$ in $D$, $w = \Phi(z)$ maps $D$ conformally onto the multiply-covered strip $-1 < \mathrm{Re}\{w\} < 1$. We further have $\Phi(\zeta) = 0$. It is easy to see that the function $\Phi(z)$ solves our extremal problem. Since the boundary values of $U(z)$ are $-1$ or $+1$, according as the expression within the absolute value signs on the right-hand side of (8) is positive or negative, (8) may also be written

$$u_\xi(\zeta) \leq -\frac{1}{2\pi} \int_\Gamma U(z)\,[\partial^2 g(z,\zeta)/\partial n \partial \xi + \alpha(\partial g(z,\zeta)/\partial n) + \sum_{\nu=1}^{n-1} \lambda_\nu (\partial \omega_\nu(z)/\partial n)]\, ds$$

whence, by (7), $u_\xi(\zeta) \leq U_\xi(\zeta)$, which proves our assertion.

We note that

$$(13)\quad U(z)P(z) \equiv U(z)[\partial^2 g(z,\zeta)/\partial n\partial\xi + \alpha(g(z,\zeta)/\partial n) + \sum_{\nu=1}^{n-1}\partial\omega_\nu(z)/\partial n] \leq 0, \quad z \in \Gamma.$$

(13) shows that $w = \Phi(z)$ yields a $(1, n)$ conformal map of $D$ onto the strip $-1 < \mathrm{Re}\{w\} < 1$. It is well known, and easily proved, that a function of the type of $P(z)$ cannot have more than $2n$ changes of sign on $\Gamma$. (13) shows that the same is true of $U(z)$. Since each sheet of the conformal map of $D$ given by $w = \Phi(z)$ corresponds to two changes of sign of $U(z)$ on $\Gamma$, it follows that this map cannot have more than $n$ sheets. On the other hand, elementary topological considerations show that the number of these sheets cannot be smaller than $n$. It thus follows that $w = \Phi(z)$ yields a $(1, n)$ mapping of $D$ onto the strip $-1 < \mathrm{Re}\{w\} < 1$.

The inequality (13) can be brought into a different form in which its connection with the functions of class $B$ is more obvious. In view of (1), the function $F(z)$ of $B$ which corresponds to the function $\Phi(z) = U(z) + iV(z)$ of $B_R$ is given by

$$(14)\quad F(z) = \tan(\pi/4)\Phi(z).$$

On $\Gamma$, the value of $U(z)$ is either 1 or $-1$, that is, $\Phi(z)$ is of the form $\Phi(z) = \pm 1 + it$, where $t$ is a real parameter. It therefore follows from (14) by an elementary computation that $F(z)/\{1 + F^2(z)\} = \pm \frac{1}{2}\cosh t$, whence

$$(15)\quad F(z)/\{U(z)[1 + F^2(z)]\} \geq 0, \qquad z \in \Gamma.$$

If $p(z, \zeta)$ denotes the analytic function whose real part is $g(z, \zeta)$, and $w_\nu(z)$ denotes the analytic function whose real part is $\omega_\nu(z)$, it follows from the Cauchy-Riemann equations and the fact that the real parts of both $p(z, \zeta)$ and $w_\nu(z)$ are constant on the boundary components $\Gamma_\nu$, that

$$(16)\quad (\partial g(z,\zeta)/\partial n)ds = -ip'(z,\zeta)dz,\ (\partial\omega_\nu(z)/\partial n)ds = -iw'_\nu(z)dz, \quad z \in \Gamma.$$

If $P(z)$ is the function defined in (13), we have therefore

$$(17)\quad P(z)ds = -i[\partial^2 p(z,\zeta)/\partial z\partial\xi + \alpha p'(z,\zeta) + \sum_{\nu=1}^{n-1}\lambda_\nu w'_\nu(z)]dz = R(z)dz, \quad z \in \Gamma.$$

$R(z)$ is a single-valued analytic function which is regular in $D + \Gamma$, except at $\zeta$, where it is of the form $R(z) = -1/(z-\zeta)^2 - \alpha/(z-\zeta) +$ regular terms. Combining (13), (15), and (17), we obtain $iF(z)R(z)/\{1 + F^2(z)\}dz \geqq 0$, with the notation

$$(18) \qquad q(z) = -R(z)/\{1 + F^2(z)\} = 1/(z-\zeta)^2 + \alpha/(z-\zeta) + \text{regular terms},$$

this can be written

$$(19) \qquad -iF(z)q(z)dz \geq 0, \qquad z \in \Gamma.$$

The function $q(z)$ is, except for its double pole at $z = \zeta$, regular in $D + \Gamma$, since the zeros of $1 + F^2(z)$ obviously coincide with the points at which $U(z)$ changes its sign; these points, in turn, coincide with the zeros of the function $P(z)$ of (13), and therefore also with those of the function $R(z)$.

Summing up, we have the following result [4]:

*If $f(z) \in B$ and $\zeta \in D$, then $|f'(\zeta)| \leq F'(\zeta)$, where $F(z)$ yields a $(1, n)$ conformal map of $D$ onto the unit circle; there exists a function $q(z)$ which is regular in $D + \Gamma$, apart from a double pole at $z = \zeta$, such that $F(z)$ and $q(z)$ are connected by the inequality* (19).

It is worth pointing out that once the existence of two functions $F(z)$ and $q(z)$ which have the required properties and are connected by (19), is shown, the extremal property of $F(z)$ and a number of related results follow easily by a proper application of the residue theorem [4]. We also note that the constant $\alpha$ in (18) is zero; indeed, from (18), (19), the residue theorem, and the fact that $|F(z)| \equiv 1$ $(z \in \Gamma)$, it follows that

$$\alpha = \tfrac{1}{2\pi i} \int_\Gamma q(z)dz = \tfrac{1}{2\pi i} \int_\Gamma [F(z)]^* F(z)q(z)dz$$

$$= [\tfrac{1}{2\pi i} \int_\Gamma F^2(z)q(z)dz]^*,$$

and this vanishes by Cauchy's theorem, since $F(\zeta) = 0$.

**3. Bounds for higher derivatives.** In view of the relation (1), the problem of maximizing $|f'(\zeta)|$ $(f(z) \in B)$ and that of maximizing $|\phi'(\zeta)|$ $(\phi(z) \in B_R)$ are one and the same thing. However, this is clearly not the case if we consider the corresponding extremal problem for higher derivatives of $f(z)$ or $\phi(z)$, and we are thus faced with two different types of problems. We shall here confine ourselves to the class $B_R$; the class $B$ presents some additional difficulties which seem beyond the reach of the method of this paper.

The domain of variablility of $\phi'(\zeta)$ $(\phi(z) \in B_R, \zeta \in D)$ is a circle. This, however, is not true any more in the case of higher derivatives. In order to

find a characterization of all points on the boundary of the domain of variability of $\phi^{(k)}(\xi)$, we therefore have to replace the problem $|\phi^{(k)}(\zeta)| = \max.$ by the more general problem

$$\text{(20)} \qquad \mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} = \max., \qquad \phi(z) \in B_R, \zeta \in D,$$

where $\theta$ is an arbitrary fixed angle.

Let $q(z,\zeta) = N(z,\zeta) + iM(z,\zeta)$ denote the analytic function whose real part is the Neumann's function $N(z,\zeta)$ of $D$. On $\Gamma$, we have $\partial N/\partial n = -2\pi/l$, where $l$ is the length of $\Gamma$. If $v(z)$ is harmonic in $D$, an application of Green's formula to the domain obtained from $D$ by deleting from it a small circle with the center $\zeta$ yields

$$v(\zeta) = \tfrac{1}{2\pi}\int_\Gamma N(z,\zeta)\,(\partial v(z)/\partial n)\,ds - l^{-1}\int_\Gamma v(z)\,ds.$$

If $v(z)$ is the imaginary part of the analytic function $\phi(z) = u(z) + iv(z)$, this formula can be cast into a different form. Using the Cauchy-Riemann equations and integration by parts, we obtain

$$\int_\Gamma N(z,\zeta)\,(\partial v/\partial n)\,ds = -\int_\Gamma N(z,\zeta)\,(\partial u/\partial s)\,ds = -\int_\Gamma N(z,\zeta)\,du$$

$$= \int_\Gamma u\,dN(z,\zeta) = \int_\Gamma u(\partial N(z,\zeta)/\partial s)\,ds = -\int_\Gamma u(\partial M(z,\zeta)/\partial n)\,ds.$$

Hence,

$$\text{(21)} \qquad v(\zeta) = -\tfrac{1}{2\pi}\int_\Gamma u(z)\,(\partial M(z,\zeta)/\partial n)\,ds - l^{-1}\int_\Gamma v(z)\,ds.$$

Combining (3) and (21) we obtain

$$\mathrm{Re}\{e^{i\theta}\phi(\zeta)\} = \cos\theta u(\zeta) - \sin\theta v(\zeta) = -\tfrac{1}{2\pi}\int_\Gamma u(z)$$

$$[\cos\theta(\partial g(z,\zeta)/\partial n - \sin\theta(\partial M(z,\zeta)/\partial n)]\,ds + l^{-1}\sin\theta\int_\Gamma v(z)\,ds.$$

Differentiating $k$ times with respect to $\xi(\zeta = \xi + i\eta)$ and observing (5), we arrive at the identity

$$\text{(22)} \quad \mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} = -\tfrac{1}{2\pi}\int_\Gamma u(z)\,[\cos\theta(\partial^{k+1}g(z,\zeta)/\partial\xi^k\partial n)$$

$$-\sin\theta(\partial^{k+1}M(z,\zeta)/\partial\xi^k\partial n) + \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n)]\,ds,$$

where the $\lambda_\nu$ are arbitrary real parameters.

The ensuing manipulation is identical with that applied to (7). Since $\phi(z)$ is in $B_R$, we have $|u(z)| \leq 1$, whence

$$(23)\quad \mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} \leq \frac{1}{2\pi}\int_\Gamma |\cos\theta(\partial^{k+1}g(z,\zeta)/\partial\xi^k\partial n) - \sin\theta(\partial^{k+1}M(z,\zeta)/\partial\xi^k\partial n) + \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n)|\,ds = \frac{1}{2\pi}\int_\Gamma |P|\,ds.$$

$\int_\Gamma |P|\,ds$ is minimized with respect to the parameters $\lambda_\nu$. As shown before, this leads to the conditions (11). We again define the bounded harmonic function $U(z)$ which takes the boundary values 1 at points of $\Gamma$ at which $P$ is negative, and the boundary values $-1$ at points at which $P$ is positive. It follows, as before, that $U(z)$ satisfies the conditions (11′) which were shown to be equivalent to the fact that the harmonic conjugate $V(z)$ of $U(z)$ is single-valued in $D$. If the single-valued and regular analytic function $\Phi(z)$ is defined by $\Phi(z) = U(z) + iV(z)$, we thus have, in view of (22), (23), and the fact that $|P| = -PU$,

$$\mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} \leq \frac{1}{2\pi}\int_\Gamma |P|\,ds = -\frac{1}{2\pi}\int_\Gamma UP\,ds = \mathrm{Re}\{e^{i\theta}\Phi^{(k)}(\zeta)\},$$

that is,

$$(24)\qquad \mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} \leq \mathrm{Re}\{e^{i\theta}\Phi^{(k)}(\zeta)\}.$$

Since the function $\Phi(z)$ is in $B_R$, (23) shows that $\Phi(z)$ indeed solves our extremal problem.

Except for a finite number of points, the real part of $\Phi(z)$ is equal either to 1 or to $-1$ at all points of $\Gamma$, and it is bounded in $D$; it follows that $w = \Phi(z)$ maps $D$ conformally onto the multiply-covered strip $-1 < \mathrm{Re}\{w\} < 1$. In order to determine the number of coverings, we observe that $|P| = -PU$, whence

$$(25)\quad UP\,ds \equiv U(z)[\cos\theta(\partial^{k+1}g(z,\zeta)/\partial\xi^k\partial n) - \sin\theta(\partial^{k+1}M(z,\zeta)\partial\xi^k\partial n) + \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n)]ds \leq 0,$$

$z \in \Gamma$. In view of $q(z,\zeta) = N(z,\zeta) + iM(z,\zeta)$, $\partial N(z,\zeta)/\partial n = -2\pi l^{-1}$ $(z \in \Gamma)$, and the Cauchy-Riemann equations, we have, along the boundary,

$$\begin{aligned} q'(z,\zeta)\,dz = dq(z,\zeta) &= dN(z,\zeta) + i\,dM(z,\zeta) \\ &= (\partial N(z,\zeta)/\partial s)\,ds + i(\partial M(z,\zeta)/\partial s)\,ds \\ &= -(\partial M(z,\zeta)/\partial n)\,ds + i(\partial N(z,\zeta)/\partial n)\,ds \end{aligned}$$

$= -(\partial M(z,\zeta)/\partial n)\,ds - (2\pi i/l)\,ds$, that is,

$$(26) \qquad (\partial M(z,\zeta)/\partial n)\,ds = -q'(z,\zeta)\,dz - (2\pi i/l)\,ds, \qquad z \in \Gamma.$$

With the help of the identities (26) and (16), (25) can be brought into the form

$$(27) \qquad -IU(z)R(z)\,dz \geq 0, \qquad z \in \Gamma,$$

where

$$R(z) = \partial^k/\partial\xi^k[\cos\theta p'(z,\zeta) + i\sin\theta q'(z,\zeta)] + \sum_{\nu=1}^{n-1} \lambda_\nu w'_\nu(z).$$

The analytic function $R(z)$ is clearly single-valued in $D$ and its only singularity in $D+\Gamma$ is a pole at $z=\zeta$, in the neighborhood of which it has the expansion

$$(28) \qquad R(z) = -k!e^{i\theta}/(z-\zeta)^{k+1} + \text{regular terms.}$$

(27) shows that the differential $iR(z)\,dz$ is real on $\Gamma$. The total variation of $\arg\{dz\}$ on $\Gamma$ is $-2\pi(n-2)$. If $\mu_1$ denotes the number of zeros of $R(z)$ in $D$ and $\mu_2$ the number of its zeros on $\Gamma$, it follows therefore from (28) and the argument principle that $2\mu_1 + \mu_2 = 2(n-2+k+1) = 2(n+k-1)$. Hence, $\mu_2 \leq 2(n+k-1)$. On the other hand, (27) shows that each change of sign of $U(z)$ on $\Gamma$ necessarily coincides with a zero of $R(z)$. Since two such changes of sign correspond to one sheet of the multiply-covered strip, it follows that the maximum number of sheets is $n+k-1$.

We have thus proved

THEOREM I. *The extremal problem* (20) *is solved by a function* $w = \Phi(z)$ *which yields a* $(1,m)$ *conformal map of* $D$ *onto the strip* $-1 < \mathrm{Re}\{w\} < 1$, *where* $m \leq n+k-1$.

Since the domain of variability of $\phi^{(k)}(\zeta)$ $(\phi(z) \in B_R)$ is necessarily simply-connected—if $\phi_1(z)$ and $\phi_2(z)$ are in $B_R$, so is $\lambda\phi_1(z) + (1-\lambda)\phi_2(z)$, $0 \leq \lambda \leq 1$ (which shows that the domain in question is even convex)—this result may also be stated in the following form:

*If* $\phi(z)$ *is in* $B_R$ *and* $\zeta \in D$, *then all boundary points of the domain of variability of* $\phi^{(k)}(\zeta)$ *are occupied by functions which yield a* $(1,m)$ *mapping of* $D$ *onto the strip* $-1 < \mathrm{Re}\{w\} < 1$, *where* $m \leq n+k-1$.

It is clear from the proof of Theorem I that any function solving the extremal problem (20) has necessarily the mapping properties mentioned in the statement of the theorem. This, however, does not imply that the

extremal function is in all cases unique. If there are two or more different sets $\{\lambda_\nu\}$ which minimize the right-hand side of (23), this will result in two or more different extremal functions.

It is worth pointing out that the problem treated in this section also leads to a positive differential of the type (19). If $F(z)$ denotes the function of $B$ which is connected with $\Phi(z)$ by the relation (1), then $F(z)[1+F^2(z)]^{-1}$ is real on $\Gamma$ and—as shown above—its sign coincides with that of $U(z)$. If we set $R(z)/\{1+F^2(z)\} = -Q(z)$, (27) yields therefore $-iF(z)Q(z)dz \geq 0$, $z \in \Gamma$; since the zeros of $1+F^2(z)$ on $\Gamma$ coincide with part of the zeros or $R(z)$, $Q(z)$ is regular in $D+\Gamma$, with the exception of a pole of order $k+1$ at $z=\zeta$.

**4. Interpolation problems.** The problem to be treated in this section is the following: Let $\alpha_1, \cdots, \alpha_p$ be points of $D$ and let $a_1, \cdots, a_p$ be complex numbers such that there exists a function $\phi_0(z)$ of $B_R$ for which $\phi_0(\alpha_i) = a_i$, $i=1, \cdots, p$; let $B_R^{(a)}$ denote the sub-class of $B_R$ which consists of all functions $\phi(z)$ that satisfy the same interpolation conditions as $\phi_0(z)$ at the points $\alpha_i$, and let $S_k$ denote the domain of variability of $\phi^{(k)}(\zeta)$ $(\zeta \in D, \phi(z) \in B_R^{(a)})$. What can be said about the functions $\phi(z)$ which occupy the boundary points of $S_k$? The problem thus consists in characterizing the functions which solve the extremal problem

$$(29)\quad \mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} = \max., \quad \phi(\alpha_i) = a_i \qquad (i=1, \cdots, p), \phi(z) \in B_R.$$

If $k=1$, it follows from (1) that this problem is identical with the corresponding interpolation problem for the class $B$—a generalization to multiply-connected domains of the classical Pick-Nevanlinna interpolation problem [9, 10]—which was treated, by different methods, in [4].

We write $a_i = b_i + ic_i$ ($b_i, c_i$ real) and normalize the functions $\phi(z)$ by the requirement

$$(30)\qquad \sum_{i=1}^{p} c_i = 0.$$

This is possible since the addition of an imaginary constant to $\phi(z)$ does not alter the fact that $\phi(z)$ is in $B_R$. Utilizing (21), (22), and the fact that $\phi(\alpha_i) = u(\alpha_i) + iv(\alpha_i) = b_i + ic_i$, we obtain the representation

$$(31)\quad \mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} = -\tfrac{1}{2\pi}\int_\Gamma u(z)\,[\cos\theta\,(\partial^{k+1}g(z,\zeta)/\partial\xi^k\partial n)$$
$$-\sin\theta\,(\partial^{k+1}M(z,\zeta)/\partial\xi^k\partial n) + \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n) + \sum_{i=1}^{p}(\sigma_i(\partial g(z,\alpha_i)/\partial n)$$
$$+\tau_i(\partial M(z,\alpha_i)/\partial n))]\,ds - \sum_{i=1}^{p}(\sigma_i b_i + \tau_i c_i),$$

where the $\lambda_\nu$, $\sigma_i$ are arbitrary real constants, and the otherwise arbitrary real constants $\tau_i$ satisfy the condition

$$\sum_{i=1}^{p} \tau_i = 0 \tag{30'}$$

(30′) has the effect of cancelling the contributions due to the second term on the right-hand side of (21).

Since $|u(z)| \leq 1$, it follows from (31) that

$$\begin{aligned}(32)\quad &\mathrm{Re}\{e^{i\theta}\phi^{(k)}(z)\} \\ &\leq \tfrac{1}{2\pi}\int_\Gamma |\cos\theta(\partial^{k+1}g(z,\zeta)/\partial\xi^k\partial n) - \sin\theta(\partial^{k+1}M(z,\zeta)/\partial\xi^k\partial n) \\ &+ \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n) + \sum_{i=1}^{p}(\sigma_i(\partial g(z,\alpha_i)/\partial n) + \tau_i(\partial M(z,\alpha_i)/\partial n))|\, ds \\ &- \sum_{i=1}^{p}(\sigma_i b_i + \tau_i c_i) = \tfrac{1}{2\pi}\int_\Gamma |P|\, ds - \sum_{i=1}^{p}(\sigma_i b_i + \tau_i c_i).\end{aligned}$$

As in the previous sections, we now proceed to minimize the right-hand side of (32) by a suitable choice of the constants $\lambda_\nu$, $\sigma_i$, $\tau_i$. The existence of the minimum is assured; as shown by (31) and (32), the value of $\mathrm{Re}\{e^{i\theta}\phi_0^{(k)}(\zeta)\}$—where $\phi_0(z)$ is any function of $B_R^{(a)}$— furnishes a lower bound for the quantity in question, and the existence of a minimizing set $\lambda_\nu$, $\sigma_i$, $\tau_i$ follows in the same way as in a similar case in Section 2. Suppose now that the values $\lambda_\nu$, $\sigma_i$, $\tau_i$, have been so chosen as to yield the minimum of the right-hand side of (32) under the condition (30′). As in Section 2, a small variation of the $\lambda_\nu$ yields the conditions

$$\int_\Gamma |P|/P(\partial\omega_\nu/\partial n)\, ds = 0, \qquad \nu = 1, \cdots, n-1. \tag{33}$$

Next, we vary the $\sigma_i$. Replacing $\sigma_j$ by $\sigma_j + \epsilon$, we obtain

$$\begin{aligned}&\tfrac{1}{2\pi}\int_\Gamma |P|\, ds + \epsilon/2\pi\int_\Gamma |P|/P(\partial g(z,\alpha_j)/\partial n)ds + o(\epsilon) - \sum_{i=1}^{p}(\sigma_i b_i + \tau_i c_i) - \epsilon b_j \\ &\qquad \geq \tfrac{1}{2\pi}\int_\Gamma |P|\, ds - \sum_{i=1}^{p}(\sigma_i b_i + \tau_i c_i),\end{aligned}$$

whence, in view of the arbitrariness of the sign of $\epsilon$,

$$\tfrac{1}{2\pi}\int_\Gamma |P|/P(\partial g(z,\alpha_j)/\partial n)\, ds = b_j. \tag{34}$$

In varying the $\tau_i$, we have to take into account the condition (30′). Replacing, accordingly, $\tau_j$ by $\tau_j + \epsilon$ and $\tau_q$ $(q \neq j)$ by $\tau_q - \epsilon$, and observing the arbitrariness of the sign of $\epsilon$, we obtain in the same manner as before

$$(35) \quad \frac{1}{2\pi}\int_\Gamma |P|/P(\partial M(z,\alpha_j)/\partial n)\,ds - \frac{1}{2\pi}\int_\Gamma |P|/P(\partial M(z,\alpha_q)/\partial n)\,ds = c_j - c_q.$$

We now introduce the bounded harmonic function $U(z)$ whose boundary values on $\Gamma$ are $-1$ or $+1$, according as $P$ is positive or negative, respectively, that is,

$$(36) \qquad U(z) = -|P|/P, \quad z \in \Gamma (P \neq 0).$$

In terms of this function, the relations (33), (34), (35) take the form

$$(33') \qquad \int_\Gamma U(z)\,(\partial \omega_\nu(z)\partial n)\,ds = 0$$

$$(34') \qquad -\frac{1}{2\pi}\int_\Gamma U(z)\,(\partial g(z,\alpha_j)/\partial n)\,ds = b_j$$

$$(35') \quad -\frac{1}{2\pi}\int_\Gamma U(z)\,(\partial M(z,\alpha_j)/\partial n)\,ds + \frac{1}{2\pi}\int_\Gamma U(z)\,(\partial M(z,\alpha_q)/\partial n)\,ds = c_j - c_q.$$

As shown by (5′), it follows from (33′) that the harmonic conjugate $V(z)$ of $U(z)$ is single-valued in $D$. The same is therefore true of the analytic function $\phi(z) = U(z) + iV(z)$, which thus belongs to $B_R$. (34′) and (35′) are—in view of (3) and (21)—equivalent to

$$(37) \qquad U(\alpha_j) = b_j$$

and $V(\alpha_j) - V(\alpha_q) = c_j - c_q$. In view of (30), it follows from the last formula that

$$(38) \qquad pV(\alpha_j) = \sum_{q=1}^{p} V(\alpha_q) = pc_j\,.$$

The harmonic conjugate $V(z)$ of $U(z)$ is only determined up to an arbitrary constant. If this constant is so chosen that $\sum_{q=1}^{p} V(\alpha_q) = 0$, (38) becomes $V(\alpha_j) = c_j$. Combining this with (37), we finally obtain

$$\Phi(\alpha_i) = U(\alpha_j) + iV(\alpha_j) = b_j + ic_j = a_j,$$

which shows that the function $\Phi(z)$ satisfies the interpolation conditions of our problem; in view of its other properties found further above, it is therefore in $B_R^{(\alpha)}$.

The function $\Phi(z)$ solves our extremal problem; indeed, as shown by (31), (32), and (36), we have

$$\begin{aligned}\mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} &\leq \tfrac{1}{2\pi}\int_\Gamma |P|\,ds - \sum_{i=1}^{p}(\sigma_i b_i + \tau_i c_i)\\ &= -\tfrac{1}{2\pi}\int_\Gamma U(z)P\,ds - \sum_{i=1}^{p}(\sigma_i b_i + \tau_i c_i) = \mathrm{Re}\{e^{i\theta}\Phi^{(k)}(\zeta)\}.\end{aligned}$$

Before we state our results in the form of a theorem, we show that our problem is again closely connected with a certain positive differential. By (16), (26), (32), and (30′),

$$\begin{aligned}(39)\quad Pds = -i\{\partial^k/\partial\xi^k[\cos\theta p'(z,\zeta) + i\sin\theta q'(z,\zeta)] + \sum_{\nu=1}^{n-1}\lambda_\nu w'_\nu(z)\\ + \sum_{i=1}^{p}[\sigma_i p'(z,\alpha_i) - i\tau_i q'(z,\alpha_i)]\}dz \equiv iR(z)\,dz, \quad z\,\varepsilon\,\Gamma,\end{aligned}$$

where the single-valued function $R(z)$ is regular in $D+\Gamma$, with the exception of the point $z=\zeta$ where it has a pole of order $k+1$, and of the points $z=\alpha_i$, $i=1,\cdots,p$, where it has single poles. Comparison of (39) and (36) shows that we have $-iU(z)R(z)\,dz \geq 0$, $z\,\varepsilon\,\Gamma$. If we introduce the bounded function $F(z)$ which is associated with $\phi(z)$ by means of (1), it follows from the relation (15), which obviously also holds for the functions $U(z)$ and $F(z)$ of this section, that

$$-iF(z)R(z)/\{1+F^2(z)\}dz \geq 0, \qquad z\,\varepsilon\,\Gamma,$$

or, with the notation $Q(z) = R(z)/\{1+F^2(z)\}$,

$$(40)\qquad -iF(z)Q(z)\,dz \geq 0, \qquad z\,\varepsilon\,\Gamma.$$

The function $Q(z)$ has the same types of singularities as $R(z)$; the zeros of $1+F^2(z)$ on $\Gamma$ do not give rise to poles of $Q(z)$, since—by virtue of (1), (36) and (39)—they clearly coincide with part of the zeros of $R(z)$. (40) permits an easy counting of the number of sheets of the conformal image of $D$ by means of $F(z)$—or, what amounts to the same thing, by means of $\Phi(z)$. The total number of poles of $Q(z)$ is $k+p+1$, where $p$ is the number of interpolation points; the total variation of $\arg\{dz\}$ along $\Gamma$ is $-2\pi(n-2)$. Since the argument of the left-hand side of (40) is constant on $\Gamma$, it therefore follows from the argument principle that the total number of zeros of

$F(z)Q(z)$ in $D+\Gamma$ (zeros on $\Gamma$ to be counted with half their multiplicity) is $k+p+n-1$. Hence, the total number of zeros of $F(z)$ in $D$ is smaller than or equal to $k+p+n-1$. Since $|F(z)| \equiv 1$ on $\Gamma$, this number is identical with the number of times the unit circle is covered by the values of $F(z)$ for $z \in D$.

We have thus proved the following theorem:

THEOREM II. *Let $\alpha_1, \cdots, \alpha_p$ be points of $D$ and let $a_1, \cdots, a_p$ be complex numbers—which, for convenience, are normalized by the requirement $\mathrm{Im}\{a_1 + \cdots + a_p\} = 0$—such that there exists a function $\phi_0(z)$ of $B_R$ for which $\phi_0(\alpha_i) = a_i$, $i = 1, \cdots, p$; if $B_R^{(a)}$ denotes the sub-class of $B_R$ which consists of all functions $\phi(z)$ which satisfy the same interpolation conditions as $\phi_0(z)$ at the points $\alpha_i$, then*

$$\mathrm{Re}\{e^{i\theta}\phi^{(k)}(\zeta)\} \leqq \mathrm{Re}\{e^{i\theta}\Phi^{(k)}(\zeta)\}, \quad \zeta \in D, \quad 0 \leqq \theta < 2\pi, \quad \Phi(z) \subset B_R^{(a)},$$

*where $w = \Phi(z)$ yields a $(1, m)$ conformal mapping of $D$ onto the strip $-1 < \mathrm{Re}\{w\} < 1$, and $m \leq k+p+n-1$. The bounded function $F(z)$, which is associated with $\Phi(z)$ by means of* (1), *satisfies the relation* (40), *where the function $Q(z)$ is single-valued and regular in $D$, with the exception of a pole of order $k+1$ at $z = \zeta$ and of $p$ simple poles at the points $\alpha_i$ $(i = 1, \cdots, p)$.*

We remark that by exactly the same method of proof we may treat the more general problem

$$\mathrm{Re}\{\sum_{j=1}^{p} \rho_j \phi^{(k_j)}(\zeta_j)\} = \max., \quad \phi(z) \in B_R^{(a)}, \quad \zeta_j \in D, \quad j = 1, \cdots, r,$$

where the $\rho_j$ are given complex numbers. The function solving this problem will again have the same general characteristics as $\Phi(z)$, the only difference being that the number of sheets of the extremal mapping may now increase to $p+n+k_1+\cdots+k_r-1$.

It has already been pointed out above that, in the case $k=1$, the interpolation problems for the classes $B$ and $B_R$, respectively, are identical. In this connection it is worth noting that the extremal property of the function $F(z)$ is an immediate consequence of the inequality (40). We recall that the function $Q(z)$ in (40) has simple poles—with the residues $\eta_i$, say—at the points $\alpha_i$ and a double pole at $z = \zeta$; if for the sake of simplicity, we add the interpolation condition $\phi(\zeta) = 0$—or, what amounts to the same thing, $f(\zeta) = 0$—the principal part of $Q(z)$ at $z = \zeta$ is easily found to be

of the form $e^{i\theta}(z-\zeta)^{-2}$. Let now $f(z)$ be in $B$ and let $f(\alpha_i) = A_i$, $i = 1, \cdots, p$, $f(\zeta) = 0$. The function $F(z)$ in (40) satisfies the same interpolation conditions, and we have $|F(z)| \equiv 1$, $z \in \Gamma$. Using this fact, the residue theorem, and (40), we obtain

$$\begin{aligned}
&\mathrm{Re}\{e^{i\theta}f'(\zeta) + \sum_{i=1}^{p} \eta_i A_i\} = \mathrm{Re}\{\tfrac{1}{2\pi i}\int_\Gamma Q(z)f(z)\,dz\} \\
&\leq \tfrac{1}{2\pi}\int_\Gamma |Q(z)\,dz| = \tfrac{1}{2\pi}\int_\Gamma |F(z)Q(z)\,dz| = \tfrac{1}{2\pi i}\int_\Gamma F(z)Q(z)\,dz \\
&= \mathrm{Re}\{\tfrac{1}{2\pi i}\int_\Gamma F(z)Q(z)\,dz\} = \mathrm{Re}\{e^{i\theta}F'(\zeta) + \sum_{i=1}^{p} \eta_i A_i\},
\end{aligned}$$

whence $\mathrm{Re}\{e^{i\theta}f'(\zeta)\} \leq \mathrm{Re}\{e^{i\theta}F'(\zeta)\}$.

**5. The radius of univalence.** It was shown by Landau [7] that a function $f(z)$ which is regular and bounded in the unit circle and satisfies $|f'(0)| = a$, $0 < a < 1$, yields a schlicht mapping of the circle $|z| < \rho = a[1 + (1-a^2)^{\frac{1}{2}}]^{-1}$. The function $f(z)$ with the smallest radius of univalence—for which this estimate is sharp—yields a $(1, 2)$ mapping of the unit circle upon itself. The objective of this section is to generalize this result to functions of class $B$ which are defined in an arbitrary multiply-connected domain. Again, we shall carry through the proofs for functions of the class $B_R$; obviously, the problems for the classes $B$ and $B_R$, respectively, are identical.

A further remark is appropriate. In the case in which $D$ reduces to the unit circle and the normalization condition refers to its center, it is natural to consider regions of univalence which are concentric circles. In the case of a general multiply-connected domain, however, it might be desirable to consider regions of univalence of a different type. For example, the concentric circles of the unit circle problem may be generalized in a natural way to regions bounded by level curves of the Green's function of $D$ with its singularity at the point of normalization. The results of this section will include these more general cases.

Let now $\phi(z)$ be a function of $B_R$, and let $\phi'(\zeta) = A = a + ib$ $(\zeta \in D)$ be given; $A$, of course, has to be an admissible value for $\phi'(\zeta)$ in accordance with the result proved in Section 2. If $z_1$ and $z_2$ are two different points in $D$ and $0 \leq \theta < 2\pi$, we have, by (3), (5), and (21),

$$(41)\quad \mathrm{Re}\{e^{i\theta}[\phi(z_1)-\phi(z_2)]\}$$
$$=-\tfrac{1}{2\pi}\int_\Gamma u(z)\{\cos\theta[\partial g(z,z_1)/\partial n-\partial g(z,z_2)/\partial n]$$
$$-\sin\theta[\partial M(z,z_1)/\partial n-\partial M(z,z_2)/\partial n]+\sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n)\}ds,$$

where $\phi(z)=u(z)+iv(z)$, and the $\lambda_\nu$ are arbitrary real parameters. In order to carry the condition that $\phi'(\zeta)=A$, we observe that, in view of (3),

$$A=a+ib=\phi'(\zeta)=\partial u(\zeta)/\partial\xi-i\partial u(\zeta)/\partial\eta$$
$$=-\tfrac{1}{2\pi}\int_\Gamma u(z)[\partial^2 g(z,\zeta)/\partial\xi\partial n-i(\partial^2 g(z,\zeta)/\partial\eta\partial n)\,ds.$$

(41) can therefore be replaced by

$$(42)\quad \mathrm{Re}\{e^{i\theta}[\phi(z_1)-\phi(z_2)]\}=-\tfrac{1}{2\pi}\int_\Gamma u(z)[\cos\theta(\partial g(z;z_1,z_2)/\partial n)$$
$$-\sin\theta(\partial M(z;z_1,z_2)/\partial n)+\sum_{\nu=1}^{n-1}\lambda_\nu\,\partial\omega_\nu/\partial n)+\alpha(\partial^2 g(z,\zeta)/\partial\xi\partial n)$$
$$+\beta(\partial^2 g(z,\zeta)/\partial\xi\partial n)\,ds]-\alpha a+\beta b,$$

where $\alpha$ and $\beta$ are arbitrary real parameters, and the meaning of the abbreviations $g(z;z_1,z_2)$ and $M(z;z_1,z_2)$ is obvious. Since $|u(z)|\leq 1$, it follows from (42) that

$$(43)\quad -|\phi(z_1)-\phi(z_2)|<\mathrm{Re}\{e^{i\theta}[\phi(z_1)-\phi(z_2)]\}$$
$$\leq\tfrac{1}{2\pi}\int_\Gamma|\cos\theta(\partial g(z;z_1,z_2)/\partial n)-\sin\theta(\partial M(z;z_1,z_2)/\partial n)$$
$$+\sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n)+\alpha(\partial^2 g(z,\zeta)/\partial\xi\partial n)+\beta(\partial^2 g(z,\zeta)/\partial\eta\partial n)|ds$$
$$-\alpha a+\beta b$$

or, with an obvious abbreviation,

$$(43')\qquad -|\phi(z_1)-\phi(z_2)|\leq\tfrac{1}{2\pi}\int_\Gamma|P|\,ds-\alpha a+\beta b.$$

As in similar cases further above, we now minimize the right-hand side of (43) by a suitable choice of the arbitrary parameters $\lambda_1,\cdots,\lambda_{n-1},\alpha,\beta,\theta$;

the existence of the minimum follows in the same elementary way as before. We assume that in (43)—or (43′)—the parameters have already been so chosen that they yield the minimum of the expression concerned, and we define the bounded harmonic function $U(z)$ by the boundary values

$$U(z) = -|P|/P, \quad z \in \Gamma, \quad P \neq 0, \tag{44}$$

that is, $U(z)$ takes the boundary values $-1$ or $+1$, according as $P$ is positive or negative, respectively. By an elementary computation, which is identical with those carried out in earlier similar cases and which we therefore omit, the necessary conditions for the existence of the minimum are found to be

$$\int_\Gamma U(z)\,(\partial\omega_\nu(z)/\partial n)\,ds = 0, \tag{45}$$

$$-\tfrac{1}{2\pi}\int_\Gamma U(z)\,(\partial^2 g(z,\zeta)\partial\xi\partial n)\,ds = a, \tag{46}$$

$$-\tfrac{1}{2\pi}\int_\Gamma U(z)\,(\partial^2 g(z,\zeta)\partial\eta\partial n)\,ds = -b,$$

$$-\tfrac{1}{2\pi}\int_\Gamma U(z)\,[\sin\theta(\partial g(z;z_1,z_2)/\partial n) + \cos\theta(\partial M(z;z_1,z_2)/\partial n)]\,ds = 0. \tag{47}$$

As shown in Section 2, (45) is equivalent to the fact that the harmonic conjugate $V(z)$ of $U(z)$—and therefore also the analytic function $\Phi(z) = U(z) + iV(z)$—is single-valued in $D$. (46) shows, in view of (3), that $\partial U/\partial x$ and $\partial U/\partial y$ have the values $a$ and $-b$, respectively, at the point $z = \zeta$. By the Cauchy-Riemann equations, it follows that $\Phi'(\zeta) = a + bi = A$. The condition (47) is found to be equivalent to

$$\mathrm{Im}\{e^{i\theta}[\Phi(z_1) - \Phi(z_2)]\} = 0, \tag{48}$$

if we recall the definitions of $g(z;z_1,z_2)$ and $M(z;z_1,z_2)$, and use (3) and (21). We have thus shown that for the "extremal angle" $\theta$, the expression $e^{i\theta}[\phi(z_1) - \phi(z_2)]$ is real. For our purposes, however, it is essential to show that this expression is moreover non-positive. This is done by considering the second variation of the right-hand side of (43) if $\theta$ is replaced by $\theta + \epsilon$. If the varied value of $P$ is denoted by $P^*$, an elementary computation shows that

$$\int_\Gamma |P^*|\,ds = \int_\Gamma |P|\,ds - \epsilon \int_\Gamma |P|/P[\sin\theta(\partial g(z;z_1,z_2)/\partial n) + \cos\theta(\partial M(z;z_1,z_2)/\partial n)]ds$$
$$-\tfrac{1}{2}\epsilon^2 \int_\Gamma |P|/P[\cos\theta(\partial g(z;z_1,z_2)/\partial n) - \sin\theta(\partial M(z;z_1,z_2)/\partial n)]ds + o(\epsilon^2).$$

The first variation vanishes by (47). Since $\int_\Gamma |P^*|\,ds \geq \int_\Gamma |P|\,ds$, the second variation is non-negative. In view of (44), this is equivalent to

$$\int_\Gamma U(z)[\cos\theta(\partial g(z;z_1,z_2)/\partial n) - \sin\theta(\partial M(z;z_1,z_2)/\partial n)\,ds \geq 0,$$

whence, by (3) and (21), $\mathrm{Re}\{e^{i\theta}[\Phi(z_1) - \Phi(z_2)]\} \leqq 0$. Combining this with (48), we obtain

$$e^{i\theta}[\Phi(z_1) - \Phi(z_2)] = -|\Phi(z_1) - \Phi(z_2)|. \tag{49}$$

On the other hand, it follows from (43′), (44) and (41) that

$$-|\phi(z_1) - \phi(z_2)| \leqq \mathrm{Re}\{e^{i\theta}[\Phi(z_1) - \Phi(z_2)]\}.$$

Combining this with (49), we finally obtain

$$-|\phi(z_1) - \phi(z_2)| \leqq -|\Phi(z_1) - \Phi(z_2)|.$$

We thus have the following result:

THEOREM III. *Let $\phi(z) \in B_R$, $\phi'(\zeta) = A$, $\zeta \in D$, $z_1 \in D$, $z_2 \in D$; then*

$$|\phi(z_1) - \phi(z_2)| \geqq |\Phi(z_2) - \Phi(z_2)|, \tag{50}$$

*where $\Phi(z)$ is likewise in $B_R$ and $\Phi'(\zeta) = A$; $w = \Phi(z)$ yields a $(1, m)$ conformal map of $D$ onto the strip $-1 < \mathrm{Re}\{w\} < 1$, where $n \leq m \leq n + 2$.*

The limitations for $m$ follow, in the same manner as on previous occasions, from (44) and the easily demonstrable fact that the function $P$ cannot have more than $2(n+2)$ changes of sign on $\Gamma$.

An immediate application of Theorem III is

THEOREM IV. *Let $f(z) \in B$ and $f'(\zeta) = A\,(\zeta \in D)$; let $\rho$ be the largest number such that all functions $\phi(z)$ with these properties are univalent in $|z - \zeta| < \rho$. Then there exists a function $F(z) \in B$, with $F'(\zeta) = A$ which*

*yields a* $(1, m)$ *map* $(n \leq m \leq n+2)$ *of* $D$ *onto the unit circle, such that* $\Phi(z)$ *is not univalent in any circle* $|z-\zeta| < \rho'$ *with* $\rho' > \rho$.

*Remark.* If $D$ is the unit circle and $\zeta$ its center, the function $F(z)$ of Theorem IV yields a $(1, m)$ mapping of the unit circle onto itself, where $m \leq 3$. According to the theorem of Landau quoted above, $m = 2$. Whether it is always true that $m \leq n+1$, or whether this is a special property of the circle, is an open question.

*Proof.* In view of the schlicht transformation (1), Theorem IV is equivalent to the corresponding result for functions $\phi(z)$ of $B_R$ with a given derivative $A$ at $z = \zeta$. It follows from Landau's theorem, or else by a standard argument using the fact that this sub-class $B_R{}^A$ of $B_R$ is compact, that there exist values $\rho > 0$ such that all such functions $\phi(z)$ are univalent for $|z-\zeta| < \rho$. Let now $\rho$ be the largest value with this property, i. e. let $\rho$ be the radius of univalence of $B_R{}^A$. A compactness argument shows that there exists a function of $B_R{}^A$, say $\phi_0(z)$, which is univalent in $|z-\zeta| < \rho$ but in no larger circle about $\zeta$. Clearly, there are two possibilities: either there are two different points $z_1$ and $z_2$ on the circumference $|z-\zeta| = \rho$ for which $\phi_0(z_1) = \phi_0(z_2)$, or else there exists a point $z_3$ on this circumference at which $\phi'_0(z_0) = 0$. In the first case, Theorem III shows that there exists a function $\Phi(z)$ in $B_R{}^A$ of the type described in the statement of Theorem IV, such that also $\Phi(z_1) = \Phi(z_2)$. It follows that $\Phi(z)$ cannot be univalent in a circle about $\zeta$ of radius larger than $\rho$. In the second case, we replace (50) by

$$|\phi(z_1) - \phi(z_2)| / |z_1 - z_2| \geq |\Phi(z_1) - \Phi(z_2)| / |z_1 - z_2|$$

and let both $z_1$ and $z_2$ tend to $z_3$. This leads to

$$|\phi'(z_3)| \geq |\Phi'_0(z_3)|, \tag{51}$$

where $\Phi_0(z)$ is the limit of a sequence of functions, all of which map $D$ onto strips whose maximum number of sheets is $n+2$. Since the class of these mappings is compact, $\Phi_0(z)$ yields a mapping of the same type. (51) shows that if there is a function $\phi_0(z)$ of $B_R{}^A$ for which $\phi'_0(z_3) = 0$ ,the same is true of a function $\Phi_0(z)$ with the properties described in the statement of Theorem IV. This completes the proof.

The fact that the regions of univalence are circles was hardly used in the proof of Theorem IV. All we used was the fact that these circles form a continuous family of domains $D_\rho$ depending on a positive parameter $\rho$,

which grows monotonically with $\rho$ and, for $\rho = 0$, shrinks to the point $z = \zeta$. Theorem IV is therefore also true for other regions of univalence which form such a family. For example, we might choose the domains $D_\rho$ to be the regions in which $g(z, \zeta) > \rho^{-1}$, that is, regions which are bounded by the level curves of the Green's function. As already said, this would be the natural generalization of the circle of univalence defined in the case of bounded functions in the unit circle.

We finally remark that, in addition to the requirement $\phi'(\zeta) = A$, the class $B_R$ may be restricted by further conditions of an interpolatory character, and we may then ask for the exact radius of univalence of this restricted class. Combining the procedure leading to Theorem IV with some of the special features of the proof of Theorem III, one can derive the corresponding result without difficulty. If the number of interpolation conditions is $k$—a given value of the first derivative counts as two conditions, etc.—then the function $w = \Phi(z)$ which, within this restricted class, has the smallest radius of univalence will yield a $(1, m)$ mapping of $D$ onto the strip $-1 < \mathrm{Re}\{w\} < 1$, where $m \leq n + k + 2$.

**6. The radius of starlikeness and the radius of convexity.** The radius of starlikeness $\rho_s(\zeta)$ of an analytic function $f(z)$ with respect to the point $\zeta$, at which $f(\zeta) = 0$, is defined as the radius of the largest circle about $\zeta$ which is mapped by $f(z)$ onto a schlicht domain that is starlike with respect to the origin. Similarly, the radius of convexity $\rho_c(\zeta)$ is defined as the radius of the largest circle about $\zeta$ which is mapped by $f(z)$ onto a convex schlicht domain. In the case of the family of bounded functions in the unit circle with a given derivative at the origin, both the function with the smallest $\rho_s(0)$ and that with the smallest $\rho_c(0)$ are identical with the function with the smallest radius of univalence, mentioned at the beginning of the preceding section. The result for $\rho_s(0)$ is due to Dieudonné [3] and that for $\rho_c(0)$ to Cacridis-Theodorakopulos [2].

The method of the preceding section requires but little modification to yield the corresponding results regarding the radius of starlikeness and the radius of convexity for functions of $B_R$. In the case of $\rho_s(\zeta)$, we set up an expression for $\mathrm{Re}\{e^{i\theta}[\phi(z_1) - \alpha\phi(z_2)]\}$—where $\alpha > 0$ is an arbitrary parameter—corresponding to (42). The ensuing procedure is then the same as that applied to (42); it finally leads to the inequality

$$|\phi(z_1) - \alpha\phi(z_2)| \geqq |\Phi(z_1) - \alpha\Phi(z_2)| \tag{52}$$

—corresponding to (50)—where $w = \Phi(z)$ again maps $D$ onto the $m$ times covered strip $-1 < \mathrm{Re}\{w\} < 1$, and $m \leq n+2$. Let now $B_R^A$ denote the class of functions $\phi(z)$ of $B_R$ for which $\phi(\zeta) = 0$ and $\phi'(\zeta) = A$. If $\rho_s(\zeta)$ denotes the radius of starlikeness of this class with respect to $z = \zeta$, then there exist a function $\Phi_0(z)$ of $B_R^A$, and two points $z_1$ and $z_2$ on the circumference $|z - \zeta| = \rho_s(\zeta) + \epsilon (\epsilon > 0)$, such that $\phi_0(z_1) - \alpha\phi_0(z_2) = 0$ for some positive $\alpha$. By (52), there also exists a function $\Phi(z)$ with the above mentioned properties for which $\Phi(z_1) - \alpha\Phi(z_2) = 0$ and which, therefore, likewise maps $|z - \zeta| < \rho_c(\zeta) + \epsilon$ onto a non-starlike domain. Letting $\epsilon \to 0$ and observing that the class of these functions $\Phi(z)$ is compact, we obtain

THEOREM V. *Let $B_R^A$ be the class of functions $\phi(z)$ of $B_R$ for which $\phi(\zeta) = 0$ and $\phi'(\zeta) = A$, and let $\rho_s(\zeta)$ be the radius of starlikeness of $B_R^A$ with respect to $\zeta$. Then $\rho_s(\zeta)$ is the exact radius of starlikeness of a function $w = \Phi(z)$ of $B_R^A$ which yields a $(1, m)$ mapping $(m \leq n+2)$ of $D$ onto the strip $-1 < \mathrm{Re}\{w\} < 1$.*

The result concerning the radius of convexity follows in exactly the same way, the only difference being that the expression $\phi(z_1) - \alpha\phi(z_2)$ is now replaced by

$$\lambda\phi(z_1) + (1-\lambda)\phi(z_2) - \phi(z_3), \qquad 0 < \lambda < 1.$$

The vanishing of this expression for a triple of distinct points $z_1$, $z_2$, $z_3$ on a circle about $z = \zeta$, and for a certain $\lambda$, is identical with the failure of the convexity property. As a result of the fact that we have to carry the additional point $z_3$ in (41) and the ensuing manipulation, the maximum number of sheets of the extremal map is raised to $n+3$. Since the proof presents no new features, we confine ourselves to the statement of the result.

THEOREM VI. *Let $B_R^A$ be the class of functions defined in the statement of Theorem V, and let $\rho_c(\zeta)$ be the radius of convexity of $B_R^A$ with respect to $z = \zeta$. Then $\rho_c(\zeta)$ is the exact radius of convexity of a function $w = \Phi(z)$ of $B_R^A$ which yields a $(1, m)$ mapping $(m \leq n+3)$ of $D$ onto the strip $-1 < \mathrm{Re}\{w\} < 1$.*

It hardly needs to be pointed out that Theorems V and VI do not solve the corresponding problems for the class $B$; obviously, the transformation (1) does not preserve the properties of starlikeness and convexity.

**7. Other problems.** By suitable modifications of the main method of proof of the preceding sections, a great variety of related problems can be solved. In this section, we shall treat two of these problems by way of illustration of the general principles involved.

The first of these problems concerns functions $\phi(z)$ which are regular and single-valued in $D$ and for which

$$-b_\mu \leq \limsup_{z\to\Gamma_\mu} \mathrm{Re}\{\phi(z)\} - a_\mu \leq b_\mu,$$

where $a_1, \cdots, a_n$ are real constants, and $b_1, \cdots, b_n$ are positive constants. This class is a generaliaztion of the class $B_R$ discussed before, and all results concerning functions of $B_R$ which were obtained in the foregoing can be extended to this wider class. As an example, we prove

THEOREM VII. *Let $\phi(z)$ be regular and single-valued in $D$, and let*

$$-b_\mu \leq \limsup_{z\to\Gamma_\mu} \mathrm{Re}\{\phi(z)\} - a_\mu \leq b_\mu, \qquad b_\mu > 0, \tag{53}$$

*where $\Gamma_1, \cdots, \Gamma_n$ are the boundary components of $D$ and, apart from the conditions $b_\mu > 0$, the $a_\mu$ and $b_\mu$ are arbitrary real constants. Then*

$$\mathrm{Re}\{\phi'(\zeta)\} \leq \mathrm{Re}\{\Phi'(\zeta)\}, \qquad \zeta \in D, \tag{54}$$

*where $\Phi(z)$ is regular and single-valued in $D$ and satisfies*

$$|\,\mathrm{Re}\{\Phi(z)\} - a_\mu\,| = b_\mu, \qquad z \in \Gamma_\mu, \tag{55}$$

*with the possible exception of not more than $2n$ points at which the boundary values of $\mathrm{Re}\{\Phi(z)\}$ may not be defined. No value is taken by $\Phi(z)$ in $D$ more than $n$ times.*

*Proof.* By Fatou's theorem, the limits of $u(z) = \mathrm{Re}\{\Phi(z)\}$ exist almost everywhere on $\Gamma$. Green's formula is valid with these boundary values of $u(z)$, since we may approximate $\Gamma$ by a sequence of level curves of the Green's function and apply the Lebesgue theorem to the resulting sequence of integrals. If $u(z)$ $(z \in \Gamma)$ denotes these boundary values, it follows—as shown before—from Green's formula and the fact that $\phi(z)$ is single-valued in $D$ that

$$\mathrm{Re}\{\phi'(\zeta)\} = -\tfrac{1}{2\pi} \int_\Gamma u(z)\,[\partial^2 g(z,\zeta)/\partial\xi\partial n + \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu/\partial n)]\,ds,$$

where the $\lambda_\nu$ are arbitrary real parameters. This formula can also be written in the form

$$(56)\quad \mathrm{Re}\{\phi'(\zeta)\} = -\sum_{\mu=1}^{n} \frac{1}{2\pi} \int_{\Gamma_\mu} [u(z) - a_\mu]\,[\partial^2 g(z,\zeta)/\partial\xi\partial n + \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu/\partial n)]\,ds$$
$$- \sum_{\mu=1}^{n-1} (a_\mu/2\pi) \int_{\Gamma_\mu} [\partial^2 g(z,\zeta)/\partial\xi\partial n + \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu/\partial n)]\,ds.$$

It follows from (53) that

$$(57)\quad \mathrm{Re}\{\phi'(\zeta)\} \leq \sum_{\mu=1}^{n} (b_\mu/2\pi) \int_{\Gamma_\mu} |\,\partial^2 g(z,\zeta)/\partial\xi\partial n + \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu/\partial n)|\,ds$$
$$- \sum_{\mu=1}^{n} (a_\mu/2\pi) \int_{\Gamma_\mu} [\partial^2 g(z,\zeta)/\partial\xi\partial n + \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu/\partial n)]\,ds.$$

We now minimize the right-hand side of (57) with respect to the parameters $\lambda_\nu$. Differentiating the right-hand side of (57) with respect to $\lambda_\nu$, the necessary conditions to be satisfied in the case of the minimum are found to be

$$(58)\quad \sum_{\mu=1}^{n} (b_\mu/2\pi) \int_{\Gamma_\mu} |\,P\,|/P\,(\partial\omega_\nu/\partial n)\,ds - \sum_{\mu=1}^{n} (a_\mu/2\pi) \int_{\Gamma_\mu} (\partial\omega_\nu/\partial n)\,ds = 0,$$
$$\nu = 1, \cdots, n-1,$$

where

$$(59)\quad P = \partial^2 g(z,\zeta)/\partial\xi\partial n + \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu(z)/\partial n), \qquad z \in \Gamma,$$

and $\lambda, \cdots, \lambda_{n-1}$, is the minimizing set. We define the harmonic function $U(z)$ by the boundary values

$$(60)\quad U(z) = a_\mu - b_\mu(|\,P\,|/P), \qquad z \in \Gamma, \qquad P \neq 0,$$

and the requirement that $|\,U(z)\,| \leq \mathrm{Max}\,|\,a_\mu\,| + b_\mu$ $(\mu = 1, \cdots, n)$ in $D$ (This requirement is necessary since the definition (60) of the boundary values of $U(z)$ breaks down at the $2n$ possible zeros of $P$). Rewriting (58) in terms of this function $U(z)$, we obtain

$$\sum_{\mu=1}^{n} \frac{1}{2\pi} \int_{\Gamma_\mu} [a_\mu - U(z)]\,(\partial\omega_\nu/\partial n)\,ds - \sum_{\mu=1}^{n} (a_\mu/2\pi) \int_{\Gamma_\mu} (\partial\omega_\mu/\partial n)\,ds = 0,$$

whence $\int_\Gamma U(z)\,(\partial\omega_\nu/\partial n)\,ds = 0$, $\nu = 1, \cdots, n-1$, which shows that the harmonic conjugate $V(z)$ of $U(z)$ is single-valued in $D$. The analytic function $\Phi(z) = U(z) + iV(z)$ is thus single-valued in $D$; as shown by (60), it has furthermore the property (55), except at the points at which $P = 0$. The extremal property (54) follows by considering, in this order, formulas (57), (59), (60), (56).

It remains to be shown that no value is taken by $\Phi(z)$ more than $n$ times in $D$. To this end, we observe from (60) that

$$P[U(z) - a_\mu] \leq 0, \qquad z \in \Gamma_\mu, \tag{61}$$

where $P$ is defined in (59). As already pointed out on a previous occasion, a function of the type (59) cannot have more than $2n$ changes of sign on $\Gamma$. (61) and (55) show that each change of sign of $P$ corresponds to a change of the value of $U(z) - a_\mu$ from $b_\mu$ to $-b_\mu$, or vice versa. Since two such changes correspond to one sheet of the conformal map of $D$ given by $\Phi(z)$, it follows that the maximum number of sheets cannot exceed $n$, whence the assertion.

Incidentally, it is not without interest to point out that our procedure furnishes an existence proof for conformal mappings with the property (55).

The next problem concerns functions $\phi(z)$ which have a simple pole at a specified point of $D$ and satisfy boundedness conditions of the type (53) on the boundary of $D$. For brevity, we take $a_\mu = 0$, $b_\mu = 1$. In order to obtain the corresponding result for more general values of these parameters, some features of the proof of Theorem VII have to be added.

THEOREM VIII. Let

$$\phi(z) = a/(z - \zeta) + O(|z - \zeta|) \qquad (\zeta \in D) \tag{62}$$

*be single-valued and, except for the simple pole at $z = \zeta$, regular in $D$, and let*

$$\limsup_{z \to \Gamma} |\operatorname{Re}\{\phi(z)\}| \leq 1. \tag{63}$$

*Then*

$$\operatorname{Re}\{a\} \leq \operatorname{Re}\{A\}, \tag{64}$$

*where $A$ is the residue, at $z = \zeta$, of a function*

$$\Phi(z) = A/(z - \zeta) + O(|z - \zeta|) \tag{65}$$

*which is single-valued and, except at $z = \zeta$, regular in $D$, and satisfies*

$$|\operatorname{Re}\{\Phi(z)\}| = 1, \qquad z \in \Gamma, \tag{66}$$

*with the possible exception of not more than $2n$ points at which $R\{\Phi(z)\}$ may not be defined. The conformal map yielded by $w = \Phi(z)$ covers each of the half-planes $\operatorname{Re}\{w\} > 1$ and $\operatorname{Re}\{w\} < 1$ once, and it covers the strip $-1 < \operatorname{Re}\{w\} < 1$ not more than $n$ times.*

*Proof.* The remark made at the beginning of the proof of Theorem VII applies also in this case, and we may apply Green's formula to the boundary

values of $u(z) = \mathrm{Re}\{\phi(z)\}$. Using (16) and the residue theorem in order to shorten the computations, we obtain

$$
\begin{aligned}
&-\tfrac{1}{2\pi}\int_\Gamma u(z)[\alpha(\partial g(z,\zeta)/\partial n) - \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu(z)/\partial n)]\,ds \\
&\quad = -\tfrac{1}{2\pi i}\int_\Gamma u(z)[\alpha p'(z,\zeta) - \sum_{\nu=1}^{n-1}\lambda_\nu w'_\nu(z)]\,dz \\
&\quad = -\mathrm{Re}\{\tfrac{1}{2\pi i}\int_\Gamma \phi(z)[\alpha p'(z,\zeta) - \sum_{\nu=1}^{n-1}\lambda_\nu w'_\nu(z)\,dz\} \\
&\quad = \mathrm{Re}\{a[\alpha c + \sum_{\nu=1}^{n-1}\lambda_\nu w'_\nu(\zeta)]\},
\end{aligned}
$$

where $\alpha, \lambda_1, \cdots, \lambda_{n-1}$, are arbitrary real parameters and

$$c = -\lim_{z\to\zeta} p'(z,\zeta) - 1/(z-\zeta). \tag{67}$$

If the arbitrary parameters are made subject to the condition

$$\alpha c + \sum_{\nu=1}^{n-1}\lambda_\nu w'_\nu(\zeta) = 1, \tag{68}$$

the above identity takes the form

$$\mathrm{Re}\{a\} = -\tfrac{1}{2\pi}\int_\Gamma u(z)[\alpha(\partial g(z,\zeta)/\partial n) - \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n)]\,ds. \tag{69}$$

In view of (63), it follows that

$$\mathrm{Re}\{a\} \leq \tfrac{1}{2\pi}\int_\Gamma |\,\alpha(\partial g(z,\zeta)/\partial n) - \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu/\partial n)|\,ds. \tag{70}$$

Our next step is to minimize the right-hand side of (70) under the conditions (68). A formal computation shows that the necessary conditions for a minimum are

$$\tfrac{1}{2\pi}\int_\Gamma |P|/P(\partial\omega_\nu/\partial n)\,ds + \mathrm{Re}\{A w'_\nu(\zeta)\} = 0, \quad \nu = 1, \cdots, n-1 \tag{71}$$

$$\tfrac{1}{2\pi}\int_\Gamma |P|/P(\partial g(z,\zeta)/\partial n)\,ds + \mathrm{Re}\{Ac\} = 0, \tag{72}$$

where

$$P = \alpha(\partial g(z,\zeta)/\partial n) - \sum_{\nu=1}^{n-1}\lambda_\nu(\partial\omega_\nu(z)/\partial n), \qquad z \in \Gamma, \tag{72'}$$

and $A$ is a complex Lagrange multiplier

We now define the bounded harmonic function $U_1(z)$ by the boundary values

$$U_1(z) = -|P|/P, \qquad z \in \Gamma, \qquad P \neq 0. \tag{73}$$

If $V_1(z)$ denotes the harmonc conjugate of $U_1(z)$, (73) and (5′) show that (71) and (72) can be rewritten in the forms

$$\tfrac{1}{2\pi}\int_{\Gamma_\nu} dV_1 = \mathrm{Re}\{Aw'_\nu(\zeta)\} \tag{74}$$

and

$$-\tfrac{1}{2\pi}\int_{\Gamma} U_1(z)\,(\partial g(z,\zeta)/\partial n)\,ds = \mathrm{Re}\{Ac\}, \tag{75}$$

respectively.

Consider now the harmonic functions

$$\psi_1(z) = \partial g(z,\zeta)/\partial\xi, \qquad \psi_2 = -\partial g(z,\zeta)/\partial\eta, \qquad \zeta = \xi + i\eta. \tag{76}$$

Both $\psi_1(z)$ and $\psi_2(z)$ are harmonic in $D$, except at $z = \zeta$ where $\psi_1(z) - \mathrm{Re}\{(z-\zeta)^{-1}\}$ and $\psi_2(z) - \mathrm{Im}\{(z-\zeta)^{-1}\}$ are harmonic. Since $g(z,\zeta)$ vanishes for $z \in \Gamma$ and any $\zeta \in D$, both $\psi_1(z)$ and $\psi_2(z)$ have the boundary values 0 at all points of $\Gamma$. The harmonic conjugates $\psi^*_1(z)$ and $\psi^*_2(z)$ of $\psi_1(z)$ and $\psi_2(z)$, respectively, are generally not single-valued. By the Cauchy-Riemann equations and the boundary properties of $\omega_\nu(z)$, we have

$$\int_{\Gamma_\nu} d\psi^*_1 = \int_{\Gamma_\nu} (\partial\psi^*_1/\partial s)\,ds = \int_{\Gamma_\nu} (\partial\psi_1/\partial n)\,ds = \int_{\Gamma} \omega_\nu(z)\,(\partial\psi_1/\partial n)\,ds.$$

Similarly, $\int_{\Gamma_\nu} d\psi^*_2 = \int_{\Gamma} \omega_\nu(z)\,(\partial\psi_2/\partial n)\,ds$. Hence, by (76),

$$\int_{\Gamma_\nu} d\psi^*_1 = \partial/\partial\xi\Big[\int_{\Gamma} \omega_\nu(z)\,(\partial g(z,\zeta)/\partial n)\,ds\Big],$$

$$\int_{\Gamma_\nu} d\psi^*_2 = -\partial/\partial\eta\Big[\int_{\Gamma} \omega_\nu(z)\,(\partial g(z,\zeta)/\partial n)\,ds\Big],$$

whence, by Green's formula,

$$\tfrac{1}{2\pi}\int_{\Gamma_\nu} d\psi^*_1 = -\partial\omega_\nu(\zeta)/\partial\xi = -\mathrm{Re}\{w'_\nu(\zeta)\},$$

$$\tfrac{1}{2\pi}\int_{\Gamma_\nu} d\psi^*_2 = \partial\omega_\nu(\zeta)/\partial\eta = -\mathrm{Im}\{w'_\nu(\zeta)\}.$$

If $A = A_1 + iA_2$, it follows from these formulas and from (74) that

$$\int_{\Gamma_\nu} d(V_1 + A_1 d\psi^*_1 - A_2 d\psi^*_1) = 0, \qquad \nu = 1, \cdots, n-1. \tag{77}$$

We now introduce the harmonic function $U(z) = U_1(z) + A_1\psi_1(z) - A_2\psi_2(z)$, its harmonic conjugate $V(z) = V_1(z) + A_1\psi^*{}_1(z) - A_2\psi^*{}_2(z)$, and the corresponding analytic function $\Phi(z) = U(z) + iV(z)$. Since

$$A_1\psi_1(z) - A_2\psi_2(z) = A_1 \operatorname{Re}\{1/(z-\zeta)\} - A_2 \operatorname{Im}\{1/(z-\zeta)\} + \cdots$$

$= \operatorname{Re}\{A/(z-\zeta)\} + c_1 + O(|z-\zeta|)$, the function $\Phi(z)$ is of the form

$$\Phi(z) = A/(z-\zeta) + c_1 + O(|z-\zeta|). \tag{78}$$

Since both $\psi_1(z)$ and $\psi_2(z)$ vanish on $\Gamma$, $U(z)$ has the same boundary values as $U_1(z)$; hence, by (73),

$$U(z) = -|P|/P, \qquad z \in \Gamma, \qquad P \neq 0. \tag{79}$$

(77) is equivalent to $\int_{\Gamma_\nu} dV = 0$, $\nu = 1, \cdots, n-1$, which shows that $V(z)$, and therefore also the analytic function $\Phi(z)$, is single-valued in $D$. The condition (75) takes the form

$$-\tfrac{1}{2\pi}\int_\Gamma U(z)\,(\partial g(z,\zeta)/\partial n)\,ds = \operatorname{Re}\{Ac\},$$

which, in view of

$$-\tfrac{1}{2\pi}\int_\Gamma U(z)\,(\partial g(z,\zeta)/\partial n)\,ds = -\operatorname{Re}\{\tfrac{1}{2\pi i}\int_\Gamma \Phi(z)p'(z,\zeta)\,dz\}$$

$= \operatorname{Re}\{Ac + c_1\}$, shows that the real part of the constant $c_1$ in (78) vanishes. Since, so far, the function $\Phi(z)$ was only determined up to an arbitrary constant, we may choose this constant so as to make the imaginary part of $c_1$ likewise vanish. We have thus $c_1 = 0$, which shows that $\Phi(z)$ is indeed of the form (65).

(79) shows that $\Phi(z)$ has the property (66). The extremal property (64) follows by considering, in that order, formulas (70), (72′), (79), (78), (68), (69), and observing that, in (78), $c_1 = 0$. As regards the number of coverings by the conformal map of $D$ as yielded by $w = \Phi(z)$, it is clear from (66) and the fact that $\Phi(z)$ has exactly one simple pole in $D$, that all points outside the strip $-1 \leq \operatorname{Re}\{w\} \leq 1$ are covered exactly once. We further have, by (73) and (72′),

$$U(z)\,[\alpha(\partial g(z,\zeta)/\partial n) - \sum_{\nu=1}^{n-1} \lambda_\nu(\partial\omega_\nu(z)/\partial n)] \leq 0, \qquad z \in \Gamma. \tag{80}$$

It is well-known, and easily proved, that the expression in the square bracket cannot change its sign on $\Gamma$ more than $2(n-1)$ times. It follows from (80)

that this is also the maximum number of changes of sign of $U(z)$ on $\Gamma$. Each two of such changes correspond to one sheet of the conformal map of $\Phi(z)$ which covers the strip $-1 < \mathrm{Re}\{w\} < 1$. The simple pole of $\Phi(z)$ contributes one extra covering of the strip by one sheet of the map which contains the point at infinity. It follows that the total number of coverings of the strip cannot exceed $n$. This completes the proof of Theorem VIII.

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# ASYMPTOTIC IDENTITIES AMONG PERIODS OF INTEGRALS OF THE FIRST KIND.*

By P. R. GARABEDIAN.

**1. Introduction.** With each closed Riemann surface $S$ of genus $g$ there are associated $g$ normal integrals of the first kind, $w_1(z), w_2(z), \cdots, w_g(z)$. The integrals $w_\nu(z)$ have the periods $\delta_{\mu\nu}\pi i$ along one set of $g$ canonical cuts $\alpha_\mu$ on $S$, while along the remaining $g$ canonical cuts $\beta_\mu$, which intersect the $\alpha_\mu$, these integrals have the periods $a_{\mu\nu}$. It is well known that $a_{\mu\nu} = a_{\nu\mu}$ and that the real quadratic form

$$\sum_{\mu,\nu=1}^{g} x_\mu x_\nu \operatorname{Re}\{a_{\mu\nu}\}$$

is positive definite.

For $g > 1$, the closed Riemann surface $S$ depends in its conformal type upon $3g-3$ complex parameters, while there are altogether $g(g+1)/2$ different quantities $a_{\mu\nu}$. Thus it is a classical problem to determine the $\frac{1}{2}g(g+1) - (3g-3) = \frac{1}{2}(g-2)(g-3)$ identities which must hold among the periods $a_{\mu\nu}$. Only in one case is this problem solved, namely, in the case $g = 4$ of surfaces of lowest genus for which an identity can be expected. Here Schottky [6] has found the one necessary identity in terms of the Riemann theta-functions associated with $S$.

Aside from the complete treatment by Schottky of the case $g = 4$, there is a paper [5] of Poincaré in which an identity among the $a_{\mu\nu}$ is found for the asymptotic situation where the diagonal terms $a_{\nu\nu}$ remain finite (different from zero), while the $a_{\mu\nu}$ with $\mu \neq \nu$ tend towards zero. Poincaré's identity involves six of the $a_{\mu\nu}$ with $\mu \neq \nu$ and has the form

$$(a_{12}a_{13}a_{24}a_{34})^{\frac{1}{2}} + (a_{13}a_{14}a_{24}a_{23})^{\frac{1}{2}} = (a_{12}a_{14}a_{23}a_{34})^{\frac{1}{2}}, \tag{1}$$

where we neglect terms of higher order.

With each plane domain $D$ of finite connectivity $n$ bounded by proper continua $C_1, \cdots, C_n$ we can associate in a unique manner a symmetric closed Riemann surface $S$ of genus $g = n-1$. The surface $S$ can be obtained, for example, as follows. We map $D$ conformally and $(n, 1)$ upon

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the interior of the unit circle so that each continuum $C_\mu$ corresponds to the unit circle traversed once. We then invert the image of $D$ in the unit circle, and we identify corresponding pairs of boundary points of the two domains so obtained. We thus construct a model with $n = g + 1$ sheets of the required closed Riemann surface $S$.

If $g = n - 1$ of the curves $C_\mu$ on $S$ are chosen to be the canonical cuts $\beta_\mu$ on $S$, then the harmonic functions

$$\omega_\mu(z) = \mathrm{Re}\{-2w_\mu(z)i/\pi\}$$

are characterized (modulo 2) by the property that they have boundary values $\delta_{\mu\nu}$ on the $C_\nu$ and are single-valued in $D$. Hence the periods

$$p_{\mu\nu} = (1/2\pi) \oint_{C_\nu} (\partial\omega_\mu/\partial n)\, ds$$

of the harmonic conjugates of the functions $\omega_\mu(z)$ are related to the quantities $a_{\mu\nu}$ according to the rule $p_{\mu\nu} = a_{\mu\nu}/\pi^2$. Thus a special case of the identities among the $a_{\mu\nu}$ will be a set of identities among the real quantities $p_{\mu\nu}$.

In the present paper we shall develop asymptotic identities among the $p_{\mu\nu}$ for various situations in which the plane domain $D$ degenerates. As a special case, we shall arrive at Poincaré's identity among the $p_{\mu\nu}$ when suitable assumptions are made upon the limiting position of the domain $D$. Our aim throughout will be to obtain an insight into the nature of the identities among the $a_{\mu\nu}$ in general, and we shall be able to avoid all restriction upon the connectivity $n$ of the domains we consider.

Our analysis will be, on the whole, altogether elementary. We shall emphasize the simple geometric interpretation of our asymptotic relations and their connections with conformal mapping. We remark that an important aspect of our problem, independent of the above discussion of closed Riemann surfaces, arises from the attempt at interpretation of the invariants $p_{\mu\nu}$ as moduli for the conformal type of $D$ [1, 2, 3].

**2. Domains with infinitesimal boundaries.** We suppose that a domain $D$ is given which is bounded by the $n$ proper continua $C_1, \cdots, C_n$. There will be no loss of generality if we assume that the $C_\nu$ are circles with, say, centers $a_\nu$, and that $a_n = \infty$, for any domain bounded by $n$ proper continua can be mapped conformally upon such a domain. We denote the radii of the circles $C_\nu$, $\nu < n$, by $t\epsilon_\nu$, while we take the radius of $C_n$ to be $R = 1/(t\epsilon_n)$, and we attempt to express the invariants $p_{\mu\nu}$ asymptotically in terms of $t$ as $t \to 0$.

For each $\nu < n$, the harmonic measures $\omega_\nu(z)$ in $D$ have the form

$$\omega_\nu(z) = \sum_{\mu=1}^{n-1} \lambda_{\mu\nu}(\log | R^2 - \bar{a}_\mu z | - \log | R(z - a_\mu) |) + O(t),$$

for suitable constants $\lambda_{\mu\nu}$. Since $R = 1/(t\epsilon_n)$, we have therefore

$$\omega_\nu(z) = -\sum_{\mu=1}^{n-1} \lambda_{\mu\nu} \log | z - a_\mu | + \lambda_{n\nu} \log t\epsilon_n + O(t), \tag{2}$$

where $\sum_{\mu=1}^{n-1} \lambda_{\mu\nu} = 0$. The $\lambda_{\mu\nu}$ are determined by the conditions that $\omega_\nu(z)$ vanishes for $z \in C_\mu$, $\mu \neq \nu$, and has the value 1 on $C_\nu$. These conditions yield the system of linear equations

$$\sum_{\mu \neq k, \mu=1}^{n-1} \lambda_{\mu\nu} \log | a_\mu - a_k | + \lambda_{k\nu} \log t\epsilon_k - \lambda_{n\nu} \log t\epsilon_n = \delta_{k\nu}, \quad k = 1, \cdots, n-1, \tag{3}$$

since we neglect terms $O(t)$.

From (2) we obtain for the periods $p_{\mu\nu}$ the asymptotic formulas

$$p_{\mu\nu} = \lambda_{\mu\nu} + O(t),$$

while from (3) the $\lambda_{\mu\nu}$ must have the form

$$\lambda_{\mu\nu} = \sigma_{\mu\nu}/| \log t | + \rho_{\mu\nu}/| \log t |^2 + O(1/| \log t |^3).$$

Considering the terms in (3) of the lowest order, namely, of the order 0 in $1/| \log t |$, we have $\sigma_{k\nu} - \sigma_{n\nu} = \delta_{k\nu}$, $k = 1, \cdots, n-1$, while at the same time we require $\sum_{\mu=1}^{n} \sigma_{\mu\nu} = 0$. Hence $\sigma_{\mu\nu} = \delta_{\mu\nu} - n^{-1}$, $\mu, \nu = 1, \cdots, n$. Thus the first order terms for the $p_{\mu\nu}$ throw no light upon the identities among these periods, and we must investigate the terms of second order.

From the terms of order 1 in $1/| \log t |$ in (3) we find for the asymptotic terms

$$\rho_{\mu\nu} = \lim_{t \to 0} \{p_{\mu\nu} - (\delta_{\mu\nu} - n^{-1})(1/| \log t |)\} | \log t |^2 \tag{4}$$

the linear equations

$$\sum_{\mu \neq k, \mu=1}^{n-1} \sigma_{\mu\nu} \log | a_\mu - a_k | + \sigma_{k\nu} \log \epsilon_k - \sigma_{n\nu} \log \epsilon_n - \rho_{k\nu} + \rho_{n\nu} = 0,$$

$k, \nu = 1, \cdots, n-1$, and we require further, of course, $\sum_{\mu=1}^{n} \rho_{\mu\nu} = 0$. We sum the first equations on $k$ and combine with the second equation to find

$$\rho_{n\nu} = -n^{-1}\sum_{k,\mu=1,\mu\neq k}^{n-1}\sigma_{\mu\nu}\log|a_\mu - a_k| - n^{-1}\sum_{k=1}^{n-1}\sigma_{k\nu}\log\epsilon_k - n^{-2}(n-1)\log\epsilon_n$$

$$= n^{-2}\sum_{k,\mu=1,\mu\neq k}^{n-1}\log|a_\mu - a_k| - n^{-1}\sum_{k=1,k\neq\nu}^{n-1}\log|a_k - a_\nu| + n^{-2}\sum_{k=1}^{n-1}\log\epsilon_k$$

$$- n^{-1}\log\epsilon_\nu - n^{-2}(n-1)\log\epsilon_n.$$

Hence

$$\rho_{\mu\nu} = n^{-2}\sum_{j,k=1,j\neq k}^{n-1}\log|a_j - a_k| + n^{-2}\sum_{k=1}^{n}\log\epsilon_k - n^{-1}\sum_{k=1,k\neq\nu}^{n-1}\log|a_k - a_\nu|$$

$$- n^{-1}\sum_{k=1,k\neq\mu}^{n-1}\log|a_k - a_\mu| - (\delta_{\mu\nu}-1)\log|a_\mu - a_\nu| - n^{-1}\log\epsilon_\nu$$

$$- n^{-1}\log\epsilon_\mu + \delta_{\mu\nu}\log\epsilon_\mu,$$

or, setting

$$\alpha_{\mu\nu} = -n^{-1}\log|a_\mu - a_\nu|, \quad \mu\neq\nu; \qquad \alpha_{\mu\mu} = -n^{-1}\log\epsilon_\mu;$$

$$A = n^{-2}\sum_{\mu,\nu=1,\mu\neq\nu}^{n-1}\log|a_\mu - a_\nu| + n^{-2}\sum_{\mu=1}^{n}\log\epsilon_\mu,$$

we obtain

$$\rho_{\mu\nu} = A + \sum_{k=1}^{n-1}\alpha_{k\mu} + \sum_{k=1}^{n-1}\alpha_{k\nu} - n\alpha_{u\nu}. \tag{5}$$

In particular, $\rho_{\mu\nu} = \rho_{\nu\mu}$, as was to be expected.

Our object now will be to solve the equations (5) for the $\alpha_{\mu\nu}$ with $\mu\neq\nu$, to obtain identities among these purely geometrical quantities, and to express the identities, finally, in terms of the asymptotic invariants $\rho_{\mu\nu}$.

We set $\alpha_\mu = \sum_{\nu=1}^{n-1}\alpha_{\mu\nu}$. From (5) we have, summing on $\nu$ from 1 to $n-1$,

$$-\rho_{\mu n} = (n-1)A + (n-1)\alpha_\mu + \sum_{\mu,\nu=1}^{n-1}\alpha_{\mu\nu} - n\alpha_\mu$$

$= (n-1)A + \sum_{\mu,\nu=1}^{n-1}\alpha_{\mu\nu} - \alpha_\mu$. Substituting the value of $\alpha_\mu$ so obtained back into (5), we find

$$n\alpha_{\mu\nu} = A + \alpha_\mu + \alpha_\nu - \rho_{\mu\nu} = A + 2(n-1)A + 2\sum_{\mu,\nu=1}^{n-1}\alpha_{\mu\nu} + \rho_{\mu n} + \rho_{\nu n} - \rho_{\mu\nu}.$$

Noting that $\sum_{\mu,\nu=1}^{n-1}\alpha_{\mu\nu} = -nA - \alpha_{nn}$ and introducing the notation $B = A + 2\alpha_{nn}$, we obtain

$$\log|a_\mu - a_\nu| = \rho_{\mu\nu} + \sum_{k=1}^{n-1}\rho_{k\mu} + \sum_{k=1}^{n-1}\rho_{k\nu} + B, \tag{6}$$

whence $d_{\mu\nu} = |a_\mu - a_\nu|^2 = \exp[2(\rho_{\mu\nu} + \sum_{k=1}^{n-1}\rho_{k\mu} + \sum_{k=1}^{n-1}\rho_{k\nu} + B)]$.

Now it is not difficult to derive polynomial identities among the squares $d_{\mu\nu}$ of the distances between the points $a_1, \cdots, a_{n-1}$. Indeed, since the volume of the degenerate tetrahedron whose vertices are any four of the points $a_\nu$, for example, whose vertices are $a_1$, $a_2$, $a_3$ and $a_4$, is zero, we have the five-by-five determinant relation

$$\begin{vmatrix} -2\,\mathrm{Re}\,(a_\mu \bar{a}_\nu) & 1 \\ 1 & 0 \end{vmatrix} = 0, \quad \mu, \nu = 1, \cdots, 4.$$

Here the term in the upper left-hand corner of the determinant represents the four-by-four matrix of quantities $-2\,\mathrm{Re}(a_\mu \bar{a}_\nu)$, and the fifth row and fifth column are formed of units, except for the vanishing diagonal term. Multiplying the fifth row of this determinant by $|a_\mu|^2$ and adding to the $\mu$-th row, and multiplying the fifth column by $|a_\nu|^2$ and adding to the $\nu$-th column, we obtain

$$(7) \qquad \begin{vmatrix} |a_\mu - a_\nu|^2 & 1 \\ 1 & 0 \end{vmatrix} = 0, \text{ i. e., } \begin{vmatrix} d_{\mu\nu} & 1 \\ 1 & 0 \end{vmatrix} = 0, \qquad \mu, \nu = 1, \cdots, 4,$$

where $d_{\mu\mu} = 0$. Thus there is a polynomial (7) in six variables and of degree three which vanishes upon substitution of values of the six distances amongst any four of the $n-1$ points $a_1, \cdots, a_{n-1}$. Alternately, we can express this fact by saying that the $n$-by-$n$ matrix

$$\left\| \begin{matrix} d_{\mu\nu} & 1 \\ 1 & 0 \end{matrix} \right\|, \qquad \mu, \nu = 1, \cdots, n-1,$$

has rank 4.

Altogether, we obtain in this way $\frac{1}{2}(n-4)(n-3) = \frac{1}{2}(g-3)(g-2)$ independent polynomial identities among the quantities $d_{\mu\nu}$. Using (6), we can now obtain from (7) a like number of identities among the limiting invariants $\rho_{\mu\nu}$. Dividing (7) by $e^{\sigma B}$ we find

$$(8) \qquad \begin{vmatrix} \gamma_{\mu\nu} \exp\left[2\left(\rho_{\mu\nu} + \sum_{k=1}^{n-1} (\rho_{k\mu} + \rho_{k\nu})\right)\right] & 1 \\ 1 & 0 \end{vmatrix} = 0,$$

$$\gamma_{\mu\nu} = 1 - \delta_{\mu\nu} = \begin{cases} 0, & \mu = \nu \\ 1, & \mu \neq \nu, \end{cases}$$

where $\mu$, $\nu$ can run over any four distinct integers from 1 to $n-1$. These identities can also be expressed in the form

$$(9)\qquad \sum_{6}\exp[2(\rho_{ij}+2\rho_{kl}-\sum_{\mu=1}^{n-1}(\rho_{\mu i}+\rho_{\mu j}))]$$
$$+\sum_{4}\exp[2(\rho_{ij}+\rho_{jk}+\rho_{ki}-2\sum_{\mu=1}^{n-1}\rho_{\mu l})]$$
$$=\sum_{12}\exp[2(\rho_{ij}+\rho_{jk}+\rho_{kl}-\sum_{\mu=1}^{n-1}(\rho_{\mu i}+\rho_{\mu l}))],$$

where the subscripts under the summation signs indicate how many terms occur, and where the indices $i$, $j$, $k$, $l$ are always distinct and range over any one set of four integers between 1 and $n-1$. Finally, we can use the equivalent expressions

$$(10)\qquad \sum_{10}\exp[2(2\rho_{ij}+\rho_{kl}+\rho_{lm}+\rho_{mk})]$$
$$=\sum_{12}\exp[2(\rho_{ij}+\rho_{jk}+\rho_{kl}+\rho_{lm}+\rho_{mi})],$$

where $i$, $j$, $k$, $l$ and $m$ are distinct and range over any one set of five integers between 1 and $n$. Formulas (10) and (9) are the same when $m=n$, and (10) follows from (9) when we remark that the choice of $C_n$ as the circle centered at infinity is arbitrary. These identities can be summarized in the statement that the $n$-by-$n$ matrix $\| \gamma_{\mu\nu}\exp(2\rho_{\mu\nu})\|$ has rank 4.

Any of the equivalent sets of identities (8), (9) or (10) represent asymptotic relations into which the identities mentioned in Section 1 must degenerate as $t\to 0$. From another point of view, one can interpret the linear equations (3) as a parametric form of these identities which would include all the terms in $1/|\log t|$ in the asymptotic expansions of the $p_{\mu\nu}$ in terms of $t$. However, the identities (10) would appear to yield more insight into the nature of the finite identities among the $p_{\mu\nu}$, especially so in view of the occurrence of exponentials of the $\rho_{\mu\nu}$ and the consequent closeness to the known result obtained by Schottky in terms of Riemann theta-functions.

**3. Domains in the unit circle.** We proceed to discuss a new case in which the $n$-th boundary curve $C_n$ of $D$ is the unit circle $|z|=1$, while the remaining $C_\nu$ are circles of radius $t\epsilon_\nu$ about points $a_\nu$ in $|z|<1$, $\nu=1,\cdots,n-1$. In such domains we have for the harmonic measures $\omega_\nu(z)$ with $\nu<n$

$$\omega_\nu(z)=\sum_{k=1}^{n-1}\lambda_{k\nu}(\log|1-\bar{a}_k z|-\log|z-a_k|)+O(t),$$

where

$$(11)\qquad \sum_{k=1,k\neq\mu}^{n-1}\lambda_{k\nu}(\log|1-\bar{a}_k a_\mu|-\log|a_\mu-a_k|)$$
$$+\lambda_{\mu\nu}(\log(1-|a_\mu|^2)-\log t\epsilon_\mu)=\delta_{\mu\nu},\qquad \mu=1,\cdots,n-1.$$

Again $p_{\mu\nu} = \lambda_{\mu\nu} + O(t)$, $\mu, \nu = 1, \cdots, n-1$, and we have the asymptotic developments

$$\lambda_{\mu\nu} = \delta_{\mu\nu}/|\log t| + P_{\mu\nu}/|\log t|^2 + O(1/|\log t|^3).$$

From the terms in $1/|\log t|$ in (11) we obtain the linear equations

$$\sum_{k=1, k\neq\mu}^{n-1} \delta_{k\nu}(\log|1-\bar{a}_k a_\mu| - \log|a_\mu - a_k|) + \delta_{\mu\nu}(\log(1-|a_\mu|^2) - \log\epsilon_\mu) + P_{\mu\nu} = 0,$$

$\mu, \nu = 1, \cdots, n-1$. Hence for $\mu = \nu$ we get

$$P_{\mu\mu} = -\log(1-|a_\mu|^2) + \log\epsilon_\mu;$$

$$P_{\mu\nu} = \log|a_\nu - a_\mu| - \log|1-\bar{a}_\nu a_\mu|, \qquad \mu \neq \nu.$$

Thus, in particular, $P_{\mu\nu} = p_{\mu\nu}/(p_{\mu\mu}p_{\nu\nu}) + O(1/|\log t|)$, $\mu \neq \nu$, and $P_{\mu\nu} = P_{\nu\mu}$.

Note that the $P_{\mu\mu}$ are independent of the $P_{\mu\nu}$ $(\mu \neq \nu)$, since $\epsilon_\mu$ can be chosen arbitrarily. Hence identities in this case will involve only the non-diagonal terms $P_{\mu\nu}$. We shall therefore assume from this point on in writing $P_{\mu\nu}$ that $\mu \neq \nu$.

We set $c_{\mu\nu} = |(a_\mu - a_\nu)/(1-\bar{a}_\mu a_\nu)|^2 = \exp(2P_{\mu\nu})$, and we note that $c_{\mu\nu}$ is closely connected with the non-Euclidean metric in the unit circle. Indeed, $c_{\mu\nu}$ is invariant when $a_\mu$ and $a_\nu$ are subjected to a conformal transformation of the interior of the unit circle upon itself. The negative of the invariant $P_{\mu\nu}$ is nothing more than the Green's function $G(a_\mu, a_\nu)$ in the unit circle.

We determine now identities amongst the squared conformal distances $c_{\mu\nu}$. There is no loss of generality if we assume that one point $a_k = 0$, $k < n$, in view of the above remarks. We thus obtain, in the notation $d_{\mu\nu} = |a_\mu - a_\nu|^2$ of Section 2, the relations

$$(1/c_{\mu\nu}) - 1$$
$$= (1 - 2\,\mathrm{Re}(a_\mu\bar{a}_\nu) + |a_\mu|^2|a_\nu|^2)/(|a_\mu|^2 - 2\,\mathrm{Re}(a_\mu\bar{a}_\nu) + |a_\nu|^2) - 1$$
$$= (1-d_{k\mu})(1-d_{k\nu}/d_{\mu\nu}, \text{ or } d_{\mu\nu} = c_{\mu\nu}(1-d_{k\mu})(1-d_{k\nu})/(1-c_{\mu\nu}).$$

We substitute this expression in the five-by-five determinant of the identity

$$\begin{vmatrix} d_{\mu\nu} & 1 \\ 1 & 0 \end{vmatrix} = 0$$

of Section 2, we divide each of the first four rows by a suitable factor

$(1 - d_{k\mu})$ and each of the first four columns by a suitable factor $(1 - d_{k\nu})$, and we obtain

$$\begin{vmatrix} c_{\mu\nu}/(1-c_{\mu\nu}) & 1/(1-d_{k\mu}) \\ 1/(1-d_{k\nu}) & 0 \end{vmatrix} = 0.$$

But $d_{k\nu} = c_{k\nu}$, $d_{k\mu} = c_{k\nu}$, since $a_k = 0$. If the fourth row and fourth column of the last determinant correspond to the index $k$, we have

$$\begin{vmatrix} c_{\mu\nu}/(1-c_{\mu\nu}) & c_{\mu k}/(1-c_{\mu k}) & 1/(1-c_{\mu k}) \\ c_{\nu k}/(1-c_{\nu k}) & 0 & 1 \\ 1/(1-c_{\nu k}) & 1 & 0 \end{vmatrix} = 0,$$

where now the upper left-hand term stands for a three-by-three matrix, while the remaining terms in $c_{\nu k}$, $c_{\mu k}$ correspond to row- and column-vectors in 3-space. Substracting the fourth row from the fifth row and the fourth column from the fifth column, we obtain, finally,

$$\begin{vmatrix} c_{\mu\nu}/(1-c_{\mu\nu}) & 1 \\ 1 & -2 \end{vmatrix} = 0, \tag{12}$$

where, once more, the upper left-hand term represents a four-by-four matrix. Since $k$ is arbitrary, there is no restriction whatever on this identity.

If we write $c_{\mu\nu}/(1-c_{\mu\nu}) = 1/(c_{\mu\nu}^{-1} - 1)$, and note that $c_{\mu\nu} < 1$, we see that (12) is actually a polynomial identity of degree nine in six variables $1/c_{\mu\nu}$. We now substitute the values of the $c_{\mu\nu}$ in terms of the $P_{\mu\nu}$ into the purely geometrical identities (12) and obtain the relations

$$\begin{vmatrix} \gamma_{\mu\nu}/(e^{-2P_{\mu\nu}} - 1) & 1 \\ 1 & -2 \end{vmatrix} = 0, \quad \gamma_{\mu\nu} = 1 - \delta_{\mu\nu}, \tag{13}$$

which are, then, polynomial identities of degree nine, each involving six exponentials $e^{-2P_{\mu\nu}}$ with $\mu \neq \nu$. There are precisely $\frac{1}{2}(n-3)(n-4) = \frac{1}{2}(g-2)(g-3)$ independent identities (13), as is evident from our geometrical derivation.

Thus for the present limiting case, the identities among the $p_{\mu\nu}$ reduce to the relations (13). Indeed, we have

$$\begin{vmatrix} \gamma_{\mu\nu}/(\exp[-2p_{\mu\nu}/(p_{\mu\mu}p_{\nu\nu})] - 1) & 1 \\ 1 & -2 \end{vmatrix} + O(1/|\log t|) = 0. \tag{14}$$

Of course, terms of higher order can be discussed starting from the parametric form (11) of the identities.

We note in passing that many further asymptotic situations of the domain $D$ can be considered, and corresponding identities will result. For example, if one allows $k$ of the continua $C_\nu$ bounding $D$ to shrink to points, while $n-k$ remain fixed, one obtains parametric identities involving the Green's function of the domain bounded by the fixed continua. However, having outlined our method, we shall restrict ourselves here to but one more case, which is of particular interest because it leads to the identities of Poincaré [5].

**4. Poincaré's identity.** We consider a situation which is analogous to the requirement that the $a_\nu$ of Section 3 tend to $|z| = 1$ at the same rate at which the radii of the circles $C_\nu$ tend to 0. However, it is no longer convenient to use canonical domains bounded by circles in our analysis.

Let $D$ lie in the upper half-planes $\mathrm{Im}(z) > 0$, let $C_n$ be the real axis, and let the $C_\nu$ with $\nu < n$ be horizontal line segments in $\mathrm{Im}(z) > 0$. Every domain of connectivity $n$ bounded by proper continua can be mapped conformally upon such a canonical region. We suppose that the segment $C_\nu$ has length $l_\nu t$ and is at a height $h_\nu t$ above the real axis. We shall study the periods $p_{\mu\nu}$ as $t \to 0$ and as the $C_\nu$ with $\nu \leqq n-1$ shrink towards points $b_\nu$ on the real axis.

If $C$ denotes a horizonal segment of length $l$ at height one unit above the real axis, we denote by $\omega(z, l)$ that function which is harmonic in the half-plane $\mathrm{Im}(z) > 0$, except on $C$, and which assumes on $C$ the value 1, while on the real axis it vanishes. We then find for the harmonic measures $\omega_\nu(z)$ in the domain $D$ the asymptotic formulas

$$(15)\quad \omega_\nu(z) = \omega((z-b_\nu)/(h_\nu t), l_\nu/h_\nu) - \sum_{\mu=1,\mu\neq\nu}^{n-1} \omega((z-b_\mu)/(h_\mu t), l_\mu/h_\mu)\omega((b_\mu + h_\mu t i - b_\nu)/(h_\nu t), l_\nu/h_\nu) + O(t^3).$$

Thus, if we take $l = l_\nu/h_\nu$,

$$\lim_{t\to 0} p_{\nu\nu} = (1/2\pi) \oint_C [\partial\omega(z, l_\nu/h_\nu)/\partial n]\, ds = \pi_{\nu\nu}$$

is different from zero and depends only upon the ratio $l_\nu/h_\nu$. On the other hand, the $p_{\mu\nu}$ with $\mu \neq \nu$ are infinitesimal and we have

$$p_{\mu\nu} = -\pi_{\mu\mu}\,\omega((b_\mu - b_\nu)/h_\nu t) + (h_\mu/h_\nu)i, l_\nu/h_\nu) + O(t^3).$$

Let us denote by $W(z, l)$ the analytic function with

$$\mathrm{Re}\{W(z, l)\} = \omega(z, l), \qquad \lim_{z\to\infty} W(z, l) = 0.$$

We find, applying Schwarz's principle of reflection, that the function

$$F(z) = W(z, l) - p \log(z + i) + p \log(z - i)$$

is analytic and single-valued in the entire $z$-plane slit along the segment $C$ and its reflection $\tilde{C}$ in the real axis. Here

$$p = (1/2\pi) \oint_C [\partial\omega(z, l)/\partial n] ds,$$

where integration is to be carried out along both edges of the slit $C$. Clearly $F(\infty) = 0$, and we shall proceed to show that

$$\eta_1 = \lim_{z \to \infty} zF(z) = 0, \qquad \eta_2 = \lim_{z \to \infty} z^2 F(z) = 0.$$

Since $W(z, l)$ is pure imaginary for $z$ real, we see that $\eta_1$ and $\eta_2$ are pure imaginary. Hence from the residue theorem we obtain

$$\eta_1 = 1/(2\pi i) \oint_{C+\tilde{C}} \{W(z, l) - p \log(z + i) + p \log(z - i)\} dz$$

$$= 1/(2\pi i) \oint_{C+\tilde{C}} \{\omega(z, l) - p \log |z + i| + p \log |z - i|\} dx,$$

since $dy = 0$ along $C + \tilde{C}$. But since we integrate in both directions along $C$ and $\tilde{C}$, and since $\omega(z, l)$ is 1 on $C$ and $-1$ on $\tilde{C}$, we find that the integral vanishes. Thus, indeed, $\eta_1 = 0$. Furthermore, by symmetry $F(iy)$ is real, and hence, since $\eta_2$ is imaginary, it must vanish.

We now obtain, then,

$$\omega((b_\mu - b_\nu)/(h_\nu t) + (h_\mu/h_\nu)i, l_\nu/h_\nu)$$

$$= \pi_{\nu\nu}(\log |(b_\mu - b_\nu)/(h_\nu t) + (h_\mu/h_\nu)i + i|$$

$$- \log |(b_\mu - b_\nu)/(h_\nu t) + (h_\mu/h_\nu)i - i|) + O(t^3)$$

$$= (\pi_{\nu\nu}/2) \log |1 + (2i)/((b_\mu - b_\nu)/(h_\nu t) + (h_\mu/h_\nu)i - i)|^2 + O(t^3)$$

$= (2\pi_{\nu\nu} h_\mu h_\nu t^2)/(b_\mu - b_\nu)^2 + O(t^3)$. Thus

$$p_{\mu\nu} = -(2\pi_{\nu\nu}\pi_{\mu\mu} h_\nu h_\mu t^2)/(b_\mu - b_\nu)^2 + O(t^3). \tag{16}$$

We set

$$\pi_{\mu\nu} = \lim_{t \to 0} p_{\mu\nu} t^{-2} = -(2\pi_{\mu\mu}\pi_{\nu\nu} h_\mu h_\nu)/(b_\mu - b_\nu)^2. \tag{17}$$

Now if $b_i < b_j < b_k < b_l$, then clearly

$$(b_l - b_i)(b_k - b_j) + (b_j - b_i)(b_l - b_k) = (b_k - b_i)(b_l - b_j),$$

and thus, introducing the reciprocal squared distances $e_{ij} = 1/(b_i - b_j)^2$ and dividing by

$$(b_j - b_i)(b_k - b_j)(b_l - b_k)(b_k - b_i)(b_l - b_i)(b_l - b_j),$$

we obtain the geometrical relation

$$(e_{ij}e_{ik}e_{jl}e_{kl})^{\frac{1}{2}} + (e_{ik}e_{il}e_{jk}e_{jl})^{\frac{1}{2}} = (e_{ij}e_{il}e_{jk}e_{kl})^{\frac{1}{2}}. \tag{18}$$

Since each index in (18) occurs twice in each term, when we substitute $e_{ij} = -\pi_{ij}/(2\pi_{ii}\pi_{jj}h_ih_j)$, from (17), the factors $\pi_{\mu\mu}$ and $h_\mu$ divide out. Thus we obtain the $\frac{1}{2}(n-3)(n-4) = \frac{1}{2}(g-2)(g-3)$ identities

$$(\pi_{ij}\pi_{ik}\pi_{jl}\pi_{kl})^{\frac{1}{2}} + (\pi_{ik}\pi_{il}\pi_{jk}\pi_{jl})^{\frac{1}{2}} = (\pi_{ij}\pi_{il}\pi_{jk}\pi_{kl})^{\frac{1}{2}}, \tag{19}$$

found originally by Poincaré [5]. These can also be wirtten

$$(p_{ij}p_{ik}p_{jl}p_{kl})^{\frac{1}{2}} + (p_{ik}p_{il}p_{jk}p_{jl})^{\frac{1}{2}} = (p_{ij}p_{il}p_{jk}p_{kl})^{\frac{1}{2}} + O(t^5). \tag{20}$$

Thus we obtain in quite elementary fashion the result which Poincaré developed from the translational character of the manifold of zeros of the theta-functions. Of course, we are restricted in (20) to the case of our special symmetric Riemann surface $S$. Our derivation of (20) indicates, however, that our method is not limited to cases where all the elements of the matrix of the $p_{\mu\nu}$ tend to zero. Finally, note that using the approach of Section 2 we can show that the identities (19) are equivalent to the statement that the $(n-1)$-by-$(n-1)$ matrix $\| \gamma_{\mu\nu}\pi_{\mu\nu}^{-1} \|$ has rank 3. This deduction is based on the fact that the $n$-by-$n$ matrix

$$\begin{Vmatrix} b_ib_j, & 1 \\ 1\ , & 0 \end{Vmatrix}$$

has rank 3 and the fact that the quantities $h_i$ are arbitrary.

**5. An alternate set of normal integrals.** Let us return for a moment to the discussion of Section 1 and make a few remarks which have some bearing

on the question of conformal type and the interpretation of the $p_{\mu\nu}$ as conformal moduli.

Let $w_{\mu\nu}(z)$ be such a conformal mapping of the multiply-connected region $D$ upon the interior of a circular ring, cut along concentric circular slits, that $C_\mu$ and $C_\nu$ are carried into the inner and outer circles of the ring, respectively, while the remaining $C_k$ correspond to the slits. Then one sees that, for example, the functions

$$W_\mu(z) = \tfrac{1}{2}\log \omega_{\mu n}(z), \qquad \mu = 1, \cdots, n-1,$$

yield a set of $g = n-1$ independent normal integrals of the first kind on the symmetric closed Riemann surface $S$ generated by $D$. This set of integrals results when we choose the curves $C_1, \cdots, C_{n-1}$ to be the canonical cuts $\alpha_\mu$ on $S$, rather than to be the cuts $\beta_\mu$ as in Section 1. The integrals $W_\mu(z)$ have a $g$-by-$g$ period matrix $\| r_{\mu\nu} \|$ about the cuts $\beta_\nu$, now taken to intersect the $C_\nu$, and this set of quantities is given, evidently, in terms of the $p_{\mu\nu}$ by the relations

$$r_{\mu\nu} = (-1)^{\mu+\nu} \mid p_{ij} \mid / \mid p_{kl} \mid ,$$
$$k, l = 1, \cdots, g, \quad i, j = 1, \cdots, g, \quad i \neq \mu, j \neq \nu,$$

where the numerator is a $(g-1)$-by-$(g-1)$ determinant, the denominator a $g$-by-$g$ determinant. Clearly $r_{\mu\nu} = r_{\nu\mu}$, and the $r_{\mu\nu}$ are, again, a special case of the invariants $a_{\mu\nu}$.

By the procedure outlined in this paper one can derive asymptotic identities among the periods $r_{\mu\nu}$ which are analogous to (10), (13) and (19). Indeed, the calculations for the $r_{\mu\nu}$ are even easier than those we have carried out for the $p_{\mu\nu}$. For example, in the case treated in Section 3, we have immediately

$$\mathrm{Re}\{W_\mu(z)\} = \tfrac{1}{2}(\log \mid z - a_\mu \mid - \log \mid 1 - a_\mu z \mid) + \text{const.} + O(t),$$

so that

$$r_{\mu\nu} = \log \mid 1 - \bar{a}_\mu a_\nu \mid - \log \mid a_\mu - a_\nu \mid + O(t), \qquad \mu \neq \nu,$$
$$r_{\mu\mu} = \log(1 - \mid a_\mu \mid^2) - \log t\epsilon_\mu + O(t).$$

Thus our identities are in certain ways even more interesting when developed in terms of the matrix $\| r_{\mu\nu} \|$ rather than the matrix $\| p_{\mu\nu} \|$. It is for this reason that we call attention to these simple relationships here.

Note, then, that our asymptotic identities among the $r_{\mu\nu}$ in the case of

Section 3 are valid to within terms $O(t)$, and that the lowest order terms in the $r_{\mu\nu}$ $(\mu \neq \nu)$ do not tend to zero with $t$, so that the identities hold among the $r_{\mu\nu}$ themselves, except for the term $O(t)$. Thus, for example, the matrix

$$\left\| \begin{matrix} \gamma_{\mu\nu}/(e^{2r_{\mu\nu}} - 1), & 1 \\ 1 \quad , & -2 \end{matrix} \right\|$$

is of rank 4 for the domain of Section 3, if we neglect terms of order of magnitude $O(t)$. Hence we obtain to some extent further information about the identities among the $a_{\mu\nu}$ when we make the elementary change from the $p_{\mu\nu}$ to the $r_{\mu\nu}$.

For the limiting domain of Section 2, we have almost without calculation the formulas

$$W_\mu(z) = \tfrac{1}{2} \log(z - a_\mu) + \text{const.} + O(t), \qquad \mu = 1, \cdots, g,$$

whence

$$r_{\mu\nu} = -\log t\epsilon_n - \log |\, a_\mu - a_\nu \,| + O(t), \qquad \mu < \nu < n,$$

$$r_{\mu\mu} = -\log t^2 \epsilon_n \epsilon_\mu + O(t), \qquad \mu < n.$$

Setting $q_{\mu\nu} = \lim_{t \to 0} \{r_{\mu\nu} + \log t\}$, we find that our identities state that the matrix

$$\left\| \begin{matrix} \gamma_{\mu\nu} e^{-2q_{\mu\nu}}, & 1 \\ 1 \quad , & 0 \end{matrix} \right\|$$

has rank 4. This form is preferable to (8), (9) or (10) because it involves all asymptotic terms in $1/|\log t\,|$.

For the case developed in Section 4, the $r_{\mu\nu}$ and $p_{\mu\nu}$ behave alike, and nothing new is discovered.

So far we have made assumptions on the asymptotic behavior of $D$ and have deduced asymptotic identities. Let us now reverse our procedure and demonstrate that our identities hold when we merely suppose that the quantities $r_{\mu\nu}$ have a prescribed limiting behavior. This will give our identities a more complete significance. We shall restrict ourselves to the situation described in Section 3, noting that the ideas apply in general.

Let us suppose, then, that we are given a domain $D$ of connectivity $n$ about which we know only that the invariants $r_{\mu\nu}$ have the form

$$r_{\mu\nu} = -\delta_{\mu\nu} \log t + R_{\mu\nu} + O(t), \qquad \mu, \nu < n, \tag{21}$$

with $t \to 0$. We shall deduce that the matrix

$$\begin{Vmatrix} \gamma_{\mu\nu}/e^{2R_{\mu\nu}} - 1), & 1 \\ 1 & , -2 \end{Vmatrix}$$

has rank 4, thus showing that our identities (13) of Section 3 hold as soon as it is known that the $p_{\mu\nu}$ tend to zero according to the rule

$$p_{\mu\nu} = \delta_{\mu\nu}/|\log t| + P_{\mu\nu}/|\log t|^2 + O(1/|\log t|^3), \qquad t \to 0,$$

which is equivalent to (21). We shall take the $R_{\mu\nu}$ to be distinct, except for the symmetry relation $R_{\mu\nu} = R_{\nu\mu}$.

We first notice that for a suitable constant $\gamma_1$ the function

$$e^{\gamma_1} w_{1n}(z) = \exp(2\, W_1(z) + \gamma_1)$$

maps $D$ upon the interior of the unit circle slit along circular arcs about the origin of radius $e^{-r\mu_1}$, which correspond to the boundary components $C_\mu$ of $D$, $1 < \mu < n$, and with the circle of radius $e^{-r_{11}}$, which corresponds to $C_1$, deleted. We can assume, therefore, that $D$ is, in fact, this image domain. We define $s$ to be the maximum diameter of the components $C_\mu$, $\mu < n$, of the boundary of our new model of the domain $D$.

It is easily seen that $s$ tends to zero with $t$, since if this were not the case, (21) could not hold. Indeed, the annular region bounded by $C_\mu$ and $C_n$ can be mapped conformally upon a circular ring with outer radius 1 and inner radius $R_\mu < e^{-r\mu\mu}$ [3]. Hence $R_\mu = O(t)$, and therefore $s = O(t)$. It follows now readily from an application of Green's theorem to the functions

$$\mathrm{Re}\{W_\mu(z)\}, \qquad \log|z - \zeta| - \log|1 - \bar{\zeta} z|$$

that

$$\mathrm{Re}\{W_\mu(z)\} + \gamma_\mu/2 = \tfrac{1}{2}(\log|z - A_\mu| - \log|1 - \bar{A_\mu} z|) + O(t)$$

for $z$ bounded away from $C_\mu$, where $A_\mu$ is any point of $C_\mu$. Thus if $A_\nu$ is any point of $C_\nu$, we have

$$r_{\mu\nu} = \log|1 - \bar{A_\mu} A_\nu| - \log|A_\mu - A_\nu| + O(t).$$

As $t \to 0$, the points $A_\mu$ on the $C_\mu$, $\mu < n$, tend to limiting positions $a_\mu$, and we have

$$R_{\mu\nu} = \log|1 - \bar{a}_\nu a_\mu| - \log|a_\nu - a_\mu|, \qquad \mu < \nu < n.$$

Thus, indeed, our identities hold among the $R_{\mu\nu}$, as stated, and the matrix

$$\left\| \begin{matrix} \gamma_{\mu\nu}/(e^{2R_{\mu\nu}}-1), & 1 \\ 1 & , -2 \end{matrix} \right\|$$

has rank 4. Similar procedures apply to our other cases, and we see that the identities of this paper do not depend upon assumptions as to the specific nature of the multiply-connected domain $D$.

It is also clear that the $r_{\mu\nu}$ determine the limit position of $D$ uniquely up to reflection and linear fractional transformation.

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# ON THE POINT SPECTRA OF $y'' + (\lambda - q(x))y = 0$.*

By GÖRAN BORG.

**1.** In several papers Wintner, Hartman and Wallach have studied the location of the continuous and discrete spectrum of

$$(L) \qquad y'' + (\lambda - q(x))y = 0,$$

$q(x)$ being continuous for $0 \leqq x < \infty$, and [1]

$$(R_a) \qquad y(0)\cos\alpha + y'(0)\sin\alpha = 0, \qquad y(x) \subset L^2(0, \infty) \text{ for } \operatorname{Im}\lambda > 0.$$

The following is a result in the same direction.

If [2]

$$(1) \qquad \int_0^x |q(t)|\,dt \leqq k_0 \log x, \qquad x \to \infty,$$

*then every point $\lambda_n$ in the point spectrum of $(L, R_a)$ satisfies $\lambda_n < k_0^2$.*

*There are equations $(L)$, satisfying (1), such that at least one eigenvalue $\lambda = k^2$ of $(L, R_a)$ satisfies $k > k_0 - \epsilon$ ($\epsilon$ arbitrarily small).*

**2. Proof of the first statement.** Put $k^2 = \lambda\ (> 0)$. Then every solution $y(x, k)$ of $(L)$ satisfies the standard inequality [3]

$$(2) \qquad k^2y^2(x, k) + y'^2(x, k) \geqq \text{const.} \exp\left(-k^{-1}\int_0^x |q(t)|\,dt\right)$$

for some const. $> 0$. According to (1), the exponential multiplying the const. is minorized by $x^{-k_0/k}$. Hence, if $x_n$ denotes a point at which $y'(x, k) = 0$, then, according to (2),

$$y^2(x_n, k) \geqq \text{const.}\ x_n^{-k_0/k}.$$

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[1] If $(R_a)$ does not alone define an eigenvalue problem, one more condition at $x = 0$ must be given. This is not the case here, for condition (1) implies limit-point behavior (no $L^2$-solution for $k > k_0$); cf. also the general results in this direction of Hartman and Wintner, *American Journal of Mathematics*, vol. 71 (1949), p. 206.

[2] There are also connections with the paper of Hartman and Wintner in *American Journal of Mathematics*, vol. 70 (1948), pp. 295-308 concerning the non-degeneracy of the wave equation.

[3] Actually, (2) is a particular case of an inequality used by Liapounoff in case of systems; for references, cf. Picard's Traité d'Analyse, 3rd ed. (1928), vol. 3, p. 385.

Suppose now that

$$\liminf n x_n^{-1} = r > 0 \qquad (n \to \infty) \tag{3}$$

and

$$\int_{x_n-\delta}^{x_n+\delta} y^2(x,k)\,dx > C y^2(x_n,k) \tag{4}$$

($\delta$ fixed, $C$ fixed for any $n$). Then if $k \geqq k_0$,

$$\int_0^\infty y^2(x,k)\,dx = \infty. \tag{5}$$

From this the statement would follow.

In fact, (3) holds true and (4) is satisfied for a subset of indices $n = m_p$ ($p = 1, 2, 3, \cdots$), dense enough to imply (5). We begin with the proof of (3), and show that the relation holds if $x_n$ represents the $n$-th zero of $y(x,k)$. In proving $r > 0$, we can suppose $q(x) \geqq 0$; for, replacing $q(x)$ in $(L)$ by $|q(x)|$, we get a set of zeros $x'_n$ and a $\liminf = r'$ satisfying $x'_n \geqq x_n$, that is, $r' \leqq r$. In order to avoid confusion, let $\xi_p$ denote the $p$-th zero of $y(x,k)$ for $x > 0$. Since now $q(x) \geqq 0$, we have $\xi_p - \xi_{p-1} = l_p \geqq \pi k^{-1}$. Then we can apply the Hilfssatz 2 in an earlier paper of the author which here takes the following form:[4]

If $l_p = \xi_p - \xi_{p-1} \geqq \pi k^{-1}$, then

$$\int_{\xi_{p-1}}^{\xi_p} |\, y''y^{-1} + k^2 \,|\, dx \geqq k^2 l_p (1 - \pi l_p^{-1} k^{-1}).$$

From this we get $\int_{\xi_1}^{\xi_p} |\, q(t) \,|\, dt \geqq k^2(\xi_p - \xi_1 - (p-1)\pi k^{-1})$. Hence from (1), $p\xi_p^{-1} \geqq k\pi^{-1} + o(1)$, that is, $\liminf p\xi_p^{-1} \geqq k\pi^{-1}$. Hence the same inequality will hold for any function $q(x)$ satisfying (1). But $x_2 < \xi_2$, $x_3 < \xi_3, \cdots, x_p < \xi_p, \cdots$, hence $p x_p^{-1} > p\xi_p^{-1}$ (after suitable numbering), which implies (3).

The proof of (5) is now simple. In fact, (3) and (1) imply that

$$\int_{x_n-\delta}^{x_n+\delta} |\, q(t) \,|\, dt > \epsilon$$

($\delta$ fixed) will hold only for certain indices $n = m_q$ ($q = 1, 2, 3, \cdots$) such

[4] G. Borg, Arkiv för Mat. Astr. och Fys., vol. 31, no. 1 (1944), p. 10.

that $m_q < n$ implies $q = O(\log n)$. Let the indices which correspond to the reversed inequality be denoted $m_p$ $(p = 1, 2, 3, \cdots)$. Since, for $m = m_p$,

$$y(x, k) = y(x_m, k)\cos k(x - x_m) + \int_{x_m}^{x} \sin k(x - t)q(t)y(t, k)k^{-1}dt,$$

we obtain

$$y(x, k) = y(x_m, k)\cos k(x - x_m)(1 + \epsilon(x)), \qquad |\epsilon(x)| < \tfrac{1}{2}$$

in a $\delta$-neighborhood of $x_m$ $(m = m_p$, $\delta$ independent of $x$, $x$ large). Thus (4) follows for $n = m_p$ $(p = 1, 2, 3, \cdots)$ and hence, if $k \geqq k_0$,

$$\int_0^{n} y^2(x, k)dx \geqq C \sum_{m_p \leqq n} m_p^{-k_0/k} \geqq C \sum_{O(\log n)}^{n} p^{-k_0/k} \to \infty, \qquad n \to \infty.$$

**3.** In order to prove the second part of the theorem, we have to construct a function $y(x, k)$ satisfying

$$\int_0^{\infty} y^2(x, k)dx < \infty \tag{6}$$

and an equation $(L)$ with $\lambda = k^2 > (k_0 - \epsilon)^2$ and with (1). This will be done with the aid of sine functions with corners, as follows. Define $y(x, k) = A_n \sin(kx - \eta_n)$ for $t_{n-1} \leqq x \leqq t_n$ and so on, so that

$$\begin{aligned} &y(t_n - 0, k) = y(t_n + 0, k) = y(t_n, k), \\ &y'(t_n + 0, k) - y'(t_n - 0, k) = y(t_n, k)\delta_n. \end{aligned} \tag{7}$$

Then $y(x, k)$ will satisfy the generalized equation $(L)$:

$$dy' + k^2 y - y dQ(x) = 0, \tag{L'}$$

where $Q(x)$ is a step-function, constant except for $x = t_n$, where $\int_{t_n} dQ(x) = \delta_n$. Let there be only one corner between two successive zeros of $y(x, k)$, denote the zeros, as above, with $\xi_n$ and let $\xi_n < t_n < \xi_{n+1}$. If $\alpha_n = k(t_n - \xi_n)$, the relations (7) give

$$\begin{aligned} A_{n+1}^2 &= A_n^2(1 + \delta_n k^{-1} \sin 2\alpha_n + \delta_n^2 k^{-2} \sin^2 \alpha_n) \\ \cot(kt_n - \eta_{n+1}) &= \cot(kt_n - \eta_n) + \delta_n k^{-1}. \end{aligned} \tag{8}$$

**4.** It is now a simple matter to construct an equation $(L')$ satisfying the relation

$$\int_0^{x} |dQ(x)| \leqq k_0 \log x$$

(corresponding to (1)), which for a $\lambda^{\frac{1}{2}} > k_0 - \epsilon$ has a solution satisfying (6). It only remains to round off the corners of the solution $y(x, k)$ in order to get an equation $(L)$ with properties required in the theorem. Put $(k_0 - \epsilon')\log x = g(x)$ $(\epsilon' < \epsilon)$ and $\delta_1 = g(t_1)$, $\delta_2 = g(t_2) - g(t_1), \cdots$, $\delta_n = g(t_n) - g(t_{n-1})$, the $t_n$ being the corners of $y(x, k)$ which we determine in the following way:

Choose $y = A_1 \sin(kx - \eta_1)$ such that $(R_\alpha)$ is satisfied for $x = 0$. If $\xi_1$ is the first zero of $y(x, k)$ which is $> 0$, choose a $t_1$ such that $\alpha_1 = k(t_1 - \xi_1) = 3/4\pi$. Now let $y(x, k)$ be defined by the expression above up to $x = t_1$. Then apply relations (8) with the numbers $\delta_1$ and $\alpha_1$ defined above, which will give us $A_2$ and $\eta_2$, determining the function $y(x, k)$ for $x > t_1$. If $y(x, k)$ is defined up to the $n$-th zero $\xi_n$, choose $t_n$ such that $\alpha_n = k(t_n - \xi_n) = 3/4\pi$ and put $y(x, k) = A_n \sin(kx - \eta_n)$ for $t_{n-1} \leqq x \leqq \xi_n$. Let the same definition hold for $t_{n-1} \leqq x \leqq t_n$. Apply (8) with the $\delta_n$ and $\alpha_n$ defined above. This determines $y(x, k) = A_{n+1} \sin(kx - \eta_{n+1})$ for $x > t_n$. Let $\xi_{n+1}$ be the first zero of this function which is $> t_n$ and repeat the preceding procedure.

This construction implies that $t_{n+1} - t_n = O(1)$ $(n \to \infty)$, hence

$$\int_0^\infty y^2(x, k)\,dx \leqq C \sum A_n^2 (t_n - t_{n-1}) \leqq \text{const.} \sum A_n^2.$$

But according to (8) and the choice of $\delta_n$ and $\alpha_n$ we have, since $\delta_p = (k_0 - \epsilon')\log(t_p/t_{p-1}) \to 0$ when $p \to \infty$,

$$A_{n+1}^2 = \prod_1^n (1 - \delta_p k^{-1} + \delta_p^2 (2k^2)^{-1}) \leqq \text{const.} \exp\left(-k^{-1}(g(t_n) - g(t_1))\right)$$

$\leqq \text{const.}\ t_n^{-(k_0 - \epsilon')/k}$. Since $t_n - t_{n-1} = O(1)$, we have $t_n = O(n)$, which implies (6) for any $k < k_0 - \epsilon'$. Furthermore

$$(9) \qquad \int_0^x |\,dQ(x)\,| = \sum_{t_n \leqq x} \delta_n = (k_0 - \epsilon')\log t_n \leqq k_0 \log x.$$

Since $\epsilon' < \epsilon$, this is the proposition, with an equation $(L')$ instead of an equation $(L)$.

It remains to replace the $y(x, k)$, defined above, by a twice continuously differentiable function in such a way that (6) still holds (for a $k = \lambda^{\frac{1}{2}} > k_0 - \epsilon$) and the inequality of (9) turns into (1). This can be done as, for instance, in the paper of the author in vol. 71 (1949), p. 69 of this JOURNAL. The substitution

$$u = y(x)\,(\sin(kx - \eta_n))^{-1}, \qquad v = -\cot(kx - \eta_n)$$

transforms the corner between the two sine functions at $x = t_n$ into one between straight lines, the case in the paper just quoted. Only simple computations are necessary to establish that the "rounding-off," made there at one corner, say $t_n$, can be made uniformly for all $n$ (this being the only new point) and in such a way that the above requirements become satisfied.

**5.** This theorem, giving a certain "best possible" result of the penetrating of the point spectrum into the continuous spectrum of $(L, R_a)$, has also another aspect. It is a theorem on the growth of the solutions of $(L)$ under condition (1), and it is interesting to notice that equations of the kind $(L')$ have a certain "extremal" property in this connection; cf. also the papers by the author cited above.

The same remark applies to the theorem of Levinson [5] on the growth of the solutions. He obtains a constant $1/\pi$, characteristic of the growth, which in fact can be replaced by the precise constant $\frac{1}{2}$ by means of a construction similar to the one used above in Sections 3-4.

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[5] N. Levinson, *Duke Mathematical Journal*, vol. 8 (1941), pp. 1-10.

# ON COMMUTATORS OF BOUNDED MATRICES.*

By C. R. PUTNAM.

**1.** It was shown by Wintner [5] that if $A$ and $B$ are bounded Hermitian matrices then the commutator $C$ of $A$ and $B$, that is

$$C = AB - BA, \tag{1}$$

cannot be a constant multiple, other than zero, of the unit matrix; cf. also Rellich [2]. It is clear, however, from the argument given in [5], that this conclusion remains valid, provided only that $A$ and $B$ are bounded, whether or not $A$ and $B$ are Hermitian; see Appendix (i) below. This extension of the theorem was formulated and proved by Wielandt [3], using a method independent of that in [5], for the case that $A$ and $B$ are bounded operators in a linear space (not necessarily a Hilbert space) with a linear metric.

The question was raised by Professor Wintner (oral communication) whether the mere assumptions that $A$ and $B$ are bounded might imply the stronger result that the greatest lower bound of the collection of numbers $|x^* Cx|$, where the vector $x$ ranges over the Hilbert unit sphere, is zero; that is, (1) satisfies

$$\text{g. l. b.}_{\|x\|=1} |x^* Cx| = 0. \tag{2}$$

(The norm $(x^* x)^{\frac{1}{2}}$ of $x$ will be denoted by $\|x\|$; for terminology, cf. [1].) That (2) does hold in case $A$ and $B$ are finite matrices is easily verified by the following argument. Let $A$ be a finite matrix and denote its trace by $\mathrm{tr}(A)$. Since, for any two finite matrices, $A$ and $B$, of the same order, $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ and $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, it follows from (1) that $\mathrm{tr}(C) = 0$. According to Hausdorff (cf. [4], p. 34) the collection of numbers $W: x^* Cx$, where $\|x\| = 1$, is a convex set. Since each of the diagonal elements of $C$ is in the set $W$ (the $n$-th diagonal element of $C$ being equal to $x^* Cx$ if $x$ is the vector whose $n$-th component is 1 and whose remaining components are 0), it follows that $\mathrm{tr}(C)/n$ is in $W$ and consequently (2) holds.

Whether or not (2) is true for arbitrary bounded $A$ and $B$ will remain an open question. The following theorem will, however, be proved:

* Received June 14, 1950.

(*) *Let $A$ and $B$ denote bounded matrices of the Hilbert space consisting of vectors $x$. For every real number $\lambda \neq 0$, let $M(\lambda)$ denote the greatest point of the spectrum of the Hermitian matrix*

$$D(\lambda) = \lambda(AA^* - A^*A) - \lambda^{-1}(BB^* - B^*B). \tag{3}$$

*If $\eta \geqq 0$ and*

$$\eta \geqq \text{g. l. b.}_{\lambda \neq 0} M(\lambda), \tag{4}$$

*then* (1) *satisfies*

$$\text{g. l. b.}_{\|x\|=1} |x^*Cx| \leqq \tfrac{1}{2}\eta. \tag{5}$$

The following definitions will be used in what follows. A bounded matrix $B$ will be called non-singular if $B$ possesses a *unique bounded* right inverse $B^{-1}$; in which case, $B^{-1}$ is also the unique bounded left inverse of $A$. A bounded Hermitian matrix $H$ will be called definite [semi-definite] if either l. u. b. $x^*Hx < 0$ [$\leqq 0$] or g. l. b. $x^*Hx > 0$ [$\geqq 0$], where $\|x\| = 1$.

As a consequence of (*) it is possible to derive the following

COROLLARY. *If either of the matrices $AA^* - A^*A$ or $BB^* - B^*B$ is semi-definite, then* (1) *satisfies* (2).

It is clear from (3) and from the fact that $\lambda$ can be any non-zero value that is is sufficient to prove the Corollary in the case that $AA^* - A^*A$ is negative semi-definite. Let the norm $|A|$ of a bounded matrix $A$ be defined by

$$|A| = \text{l. u. b.} |y^*Ax|, \qquad \|x\| = \|y\| = 1.$$

Then (3) shows that, for $\lambda > 0$ and $\|x\| = 1$,

$$x^*D(\lambda)x \leqq -\lambda^{-1}x^*(BB^* - B^*B)x,$$

and therefore, since $|BB^* - B^*B| \leqq 2|B|^2$,

$$x^*D(\lambda)x \leqq 2|B|^2\lambda^{-1}.$$

The matrix $D(\lambda)$ is Hermitian; hence $M(\lambda) = \text{l. u. b.}\, x^*D(\lambda)x$, where $\|x\| = 1$, so that $M(\lambda) \leqq 2|B|^2\lambda^{-1}$. Since $\lambda$ is an arbitrary positive number, (4) holds for $\eta = 0$ and (2) follows from (*).

*Remark.* Clearly, if $A$ is normal, then $AA^* - A^*A$ (the zero matrix) is semi-definite. It is noteworthy that the converse of this statement is not true in general, but does hold if $A$ is a finite matrix. That is, *there do not exist finite, but there do exist bounded, matrices for which $AA^* - A^*A$ is semi-definite and $A$ is not normal*; cf. Appendix (ii).

**2.** *Proof of* (*). For any pair of real numbers $\lambda \neq 0$ and $\theta$ define the bounded matrix $F$ by

$$F = |\lambda|^{\frac{1}{2}} e^{i\theta} A + |\lambda|^{-\frac{1}{2}} B^*, \text{ so that } F^* = |\lambda|^{\frac{1}{2}} e^{-i\theta} A^* + |\lambda|^{-\frac{1}{2}} B.$$

It is easily verified that

$$FF^* - F^* F = |\lambda|(AA^* - A^* A) - |\lambda|^{-1}(BB^* - B^* B) + H, \tag{6}$$

where $H = e^{i\theta} C + e^{-i\theta} C^*$. If $D(\lambda)$ is defined by (3), relation (6) can be expressed as

$$(\operatorname{sgn}\lambda)(FF^* - F^* F) = D(\lambda) + (\operatorname{sgn}\lambda) H, \tag{7}$$

and consequently,

$$(\operatorname{sgn}\lambda) x^*(FF^* - F^* F)x = x^* D(\lambda) x + 2(\operatorname{sgn}\lambda)\Re(e^{i\theta} x^* Cx). \tag{8}$$

Suppose (*) is false; that is, suppose that there exists a real number $\eta \geqq 0$ such that (4) holds and (5) fails to hold. Thus $\tau > \frac{1}{2}\eta$, where

$$\tau = \text{g. l. b.}_{\|x\|=1} |x^* Cx|. \tag{9}$$

It is known (cf. [4], p. 34) that the closure $W$ of the collection of numbers $x^* Cx$, where $\|x\| = 1$, is a convex set. (If $W_n$ denotes the set of numbers $x^* C_n x$, where $\|x\| = 1$ and $C_n$ is the finite $n$-th section matrix of $C$, then $W_n$ is contained in $W_{n+1}$ and $W$ is the closure of the set $\Sigma W_n$.) Since $\tau > 0$, it is possible to choose real values $\theta$ so that either $\Re(e^{i\theta} x^* Cx) \geqq \tau$ or $\Re(e^{i\theta} x^* Cx) \leqq -\tau$ for all vectors $x$ satisfying $\|x\| = 1$. (The set $We^{i\theta}$ is merely the set $W$ rotated through an angle $\theta$.) In virtue of (4), it is possible to find a real number $\lambda \neq 0$ such that $M(\lambda) < 2\tau$; for this value $\lambda$, choose a real $\theta$ so that $(\operatorname{sgn}\lambda)\Re(e^{i\theta} x^* Cx) \leqq -\tau$. For the above pair $\lambda$ and $\theta$ it is seen that, since $D(\lambda)$ is Hermitian, the right side of (8) is not greater than $-2\tau + M(\lambda)$ for all vectors $x$ satisfying $\|x\| = 1$. That is, the operator $D(\lambda) + (\operatorname{sgn}\lambda) H$ is negative definite. It will now be shown that if $F$ is bounded, then $FF^* - F^* F$ cannot be definite.

It can be supposed that $F$ is non-singular (otherwise, replace $F$ by $G$, where $G$ differs from $F$ by an appropriate constant multiple of the unit matrix, and note that $GG^* - G^* G = FF^* - F^* F$). Since $FF^* = F(F^* F)F^{-1}$, the matrices $FF^*$ and $F^* F$ have the same spectrum; cf. [5].

It is seen that $x^*(FF^* - F^* F)x = x^* FF^* x - x^* F^* Fx$. Let $\epsilon$ denote an arbitrary positive number, and $\lambda_0$ the least point of the spectrum of $FF^*$ and $F^*F$. Clearly, both $FF^*$ and $F^* F$ are Hermitian. Thus there exists a vector $y$, $\|y\| = 1$, such that $y^* FF^* y < \lambda_0 + \epsilon$; moreover, $x^* F^* Fx \geqq \lambda_0$

for all $x$ with $\| x \| = 1$, and consequently $y^*(FF^* - F^* F)y < \epsilon$. Since the positive number $\epsilon$ is arbitrary, $FF^* - F^* F$ cannot be positive definite. Similar reasoning when $\lambda_0$ is the greatest point of the spectrum of $FF^*$ and $F^* F$ shows that $FF^* - F^* F$ cannot be negative definite. These results are in contradiction with (7) and the previous result that $D(\lambda) + (\operatorname{sgn} \lambda)H$ is negative definite for some real $\lambda \neq 0$. Accordingly, a contradiction to the assumption that (*) is false is obtained and the proof of (*) is complete.

## Appendix.

(i) Since (1) remains unchanged if $A$ is replaced by a matrix differing from $A$ by a constant multiple of the unit matrix, it can be supposed that $A$ is non-singular. Since $AB = A(BA)A^{-1}$, it follows that $AB$ and $BA$ have identical spectra. Suppose, if possible, that (1) is a constant multiple $\sigma \neq 0$ of the unit matrix; then the spectrum of $AB$ is the set obtained by translating the spectrum of $BA$ by $\sigma$. Hence the spectrum of $AB$ (or $BA$) is unbounded; since, however, $AB$ and $BA$ are bounded, their spectra are bounded. This contradiction shows that $C$ cannot be a non-zero constant multiple of the unit matrix; cf. [5], where the above argument is used.

(ii) Let $A$ denote a finite matrix for which the Hermitian matrix $AA^* - A^* A$ is semi-definite. Without loss of generality it can be supposed that the latter matrix is non-negative definite and that consequently its eigenvalues are non-negative. Since $\operatorname{tr}(AA^* - A^* A) = 0$, it follows that all these eigenvalues are zero. Hence $AA^* - A^* A$ is equivalent (in fact, unitarily equivalent) to the zero matrix and consequently is itself the zero matrix; that is, $A$ is normal. On the other hand, there exist bounded matrices $A$ such that $AA^* - A^* A$ is semi-definite and $A$ is not normal. Such a matrix $A = (a_{ij})$ is given by $a_{ii+1} = 1$, $a_{ij} = 0$, $j \neq i + 1$. It is easily verified that the elements of $AA^* - A^* A = (\alpha_{ij})$ are all zero except $\alpha_{11}$ $(= 1)$. Actually the matrix $A$ is singular (since $A^*$ is a right but not a left hand reciprocal) but a non-singular matrix $B$ for which $BB^* - B^* B = AA^* - A^* A$, is easily constructed; cf. (i) above.

(iii) It follows from (*) that if $\tau > 0$, where $\tau$ is defined by (9), then $\tau \leqq \frac{1}{2}$ l. u. b. $_{\|x\|=1} x^* D(\lambda)x$, for every fixed real $\lambda \neq 0$. Define $a$ and $b$ by

$$a = \text{l. u. b.}_{\|x\|=1} | x^*(AA^* - A^* A)x | \quad \text{and} \quad b = \text{l. u. b.}_{\|x\|=1} | x^*(BB^* - B^* B)x | .$$

Clearly, if $\tau \geqq 0$, then $\tau \leqq \frac{1}{2}(\lambda a + b\lambda^{-1})$ holds for all $\lambda > 0$. Suppose that $a > 0$ and $b > 0$. Then $\frac{1}{2}(\lambda a + b\lambda^{-1})$ has for $0 < \lambda < \infty$ the minimum value $(ab)^{\frac{1}{2}}$ so that

$$\tau \leqq (ab)^{\frac{1}{2}}. \tag{10}$$

If $a = 0$, it follows from the definition of $a$ that $A$ is normal; if $b = 0$, then $B$ is normal. Thus, if either $a = 0$ or $b = 0$, the Corollary of (*) shows that $\tau = 0$ so that (10) is still valid. These results may be summarized in the inequality

$$(11)\quad (\text{g. l. b.}_{\|x\|=1} \mid x^*(AB - BA)x \mid)^2 \leqq (\text{l. u. b.}_{\|x\|=1} \mid x^*(AA^* - A^*A)x \mid)(\text{l. u. b.}_{\|x\|=1} \mid x^*(BB^* - B^*B)x \mid),$$

which is valid for any pair of bounded matrices $A$ and $B$.

It remains undecided whether (11) is trivial in the sense that the left side is zero for every such pair of matrices.

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# ON THE PROBLEMS OF GEODESICS IN THE SMALL.*

By PHILIP HARTMAN and AUREL WINTNER.

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**1.** In the positive definite line element

$$ds^2 = g_{ik}du_i du_k, \qquad (u_1, u_2) = (u, v), \tag{1}$$

let the coefficient functions $g_{ik} = g_{ik}(u, v)$ be defined and of class $C^1$ in a vicinity of a point, say $(u, v) = (0, 0)$. Then the Christoffel symbols $\Gamma^i_{jk}$ exist and are continuous functions of $(u, v)$. Hence the differential equations

$$u_i'' + \Gamma^i_{jk}u_j' u_k' = 0 \tag{2}$$

are meaningful and define the geodesics belonging to (1). The term "geodesic" below will always refer to a solution of (2) (rather than to an arc minimizing the arc-length integral belonging to (1)).

The question of whether or not an initial point and direction determine a unique geodesic (locally) has been answered in the negative ([5], § 2). Despite this fact, the following positive assertion will turn out to be true:

(I) *Let* (1) *be positive definite, with coefficients* $g_{ik}(u, v)$ *of class* $C^1$, *in a vicinity of* $(u, v) = (0, 0)$. *Then, to every sufficiently small* $\epsilon > 0$, *there belongs a* $\delta > 0$ *having the property that every point* $(u, v) \neq (0, 0)$ *of the circle* $C_\delta: u^2 + v^2 \leqq \delta^2$ *can be joined with* $(0, 0)$ *by a geodesic contained in the circle* $C_\epsilon$.

**2.** The following questions immediately arise:

(i) Under the assumptions of (I), is the geodesic supplied by (I) unique?

(ii) If $(u, v) \neq (0, 0)$ is a point of $C_\epsilon$, then, according to Hilbert ([6], p. 186; cf. [3], pp. 422-427), there exists in $C_\epsilon$ a rectifiable arc joining $(u, v)$ with $(0, 0)$, for which the arc-length (in the metric (1)) is the least among all such arcs. Is a minimizing arc unique if $(u, v)$ is sufficiently near to $(0, 0)$?

(iii) Corresponding to a given direction through $(0, 0)$, are there points $(u, v)$, arbitrarily near to $(0, 0)$, such that a minimizing arc, contained in $C_\epsilon$ and joining $(0, 0)$ with $(u, v)$, has the assigned initial direction?

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(iv) Are the answers to (i), (ii) and/or (iii) affirmative when every direction at $(0, 0)$ determines a unique geodesic (locally) through $(0, 0)$?

(v) Conversely, if the answers to (i), (ii) and/or (iii) are affirmative, does the point $(0, 0)$ and a direction through $(0, 0)$ determine a unique geodesic (locally)?

(vi) Must a geodesic furnish the minimal distance (in the sense of Hilbert, cf. (ii)) between some pair of points on it?

(II) *Under the assumptions of* (I), *the answer to each of the questions* (i)-(vi) *is in the negative.*

*Remark.* It will be seen that the answers to (i)-(vi) remain negative even if the Christoffel symbols $\Gamma^i_{jk}$, instead of being just continuous as in (I), (II), are required to satisfy a uniform Hölder condition of any given order $\alpha < 1$. But if $\alpha = 1$, the situation changes entirely:

(II bis) *If, in addition to the assumptions of* (I), *the partial derivatives* $\partial g_{ik}/\partial u_j$ *are assumed to satisfy a uniform Lipschitz condition, then the answer to each of the questions* (i)-(vi) *is in the affirmative.*

**3.** More than (II bis) is contained in the following theorem:

(III) *Under the assumptions of* (II bis), *there exists, on a sufficiently small circle* $\mathfrak{k}$: $(0 \leqq r \leqq a,\ 0 \leqq \phi < 2\pi)$, *a pair of functions*

$$u = u(r, \phi), \qquad v = v(r, \phi) \tag{3}$$

*having the following properties*: *Both functions* (3) *are of class* $C^1$ *on* $\mathfrak{k}$, *vanish at the center of* $\mathfrak{k}$, *and* (3) *is, for every fixed* $\phi$, *a geodesic, on which* $r$ *is the arc-length. Furthermore,* (3) *represents a topological mapping of* $\mathfrak{k}$ *on a neighborhood of* $(u, v) = (0, 0)$ *and the Jacobian* $\partial(u, v)/\partial(r, \phi)$ *vanishes only at the center of* $\mathfrak{k}$. *Finally,* (3) *transforms* (1) *into the geodesic normal form*

$$ds^2 = dr^2 + g d\phi^2, \tag{4}$$

*where* $g = g(r, \phi)$ *is continuous on* $\mathfrak{k}$ *and is positive except at the center of* $\mathfrak{k}$.

The classical methods prove (III) under the assumption that the functions $g_{ik}$ are of class $C^2$, or even of class $C^3$, instead of being, as in (III), only of class $C^1$ with partial derivatives $\partial g_{ik}/\partial u_j$ satisfying a uniform Lipschitz condition. Correspondingly, the first point in (III) is that the fundamental

field construction, classically based on the class $C^2$ (cf. [4], pp. 408-409), can be carried out under the less severe assumptions of (II bis).

The situation goes even further in the existence proof for the Gauss normal form (4). In fact, the usual proof has to assume, to this end, that the functions $g_{ik}$ are of class $C^3$ (cf., for instance, [2], p. 101, where the deduction of (4) is based on the calculation of the Christoffel symbols in terms of the new coordinates $r, \phi$; a calculation involving a differentiation, the legitimacy of which is anything but obvious under the $C^2$-assumption).

Clearly, the deduction of (4) under the relaxed assumptions of (III) is equivalent to the deduction, under such assumptions, of Gauss's theorem on the transversals of geodesics. Correspondingly, the point will be that Gauss's theorem can be deduced under assumptions which are less severe than in the usual proofs.

**4. Proof of (I).** Let $\epsilon > 0$ be so small that the $g_{ik}$ are of class $C^1$ in the closed circle $C_\epsilon: u^2 + v^2 \leqq \epsilon^2$. Let $M$, $m$ be a pair of constants satisfying

$$M \geqq g_{ik}(u, v)\xi_i\xi_k/(\xi_1^2 + \xi_2^2) \geqq m > 0 \tag{5}$$

for every $(u, v)$ in $C_\epsilon$. Choose $\delta$ so small that $m\epsilon > M\delta$. If $(u, v)$ is a point of $C_\delta$, let $\Gamma = \Gamma(u, v)$ be a rectifiable arc in $C_\epsilon$ joining $(0, 0)$ with $(u, v)$ and minimizing the arc-length, in the metric (1), among all such arcs. Then no point of $\Gamma$ is on the boundary of $C_\epsilon$. For the length of the arc joining $(0, 0)$ with a point of the boundary of $C_\epsilon$ is at least $m\epsilon$, while the length of the line segment joining $(0, 0)$ with $(u, v)$ is at most $M\delta < m\epsilon$.

In order to prove (I), it will be shown that $\Gamma$ is a geodesic (which, in view of (2), implies that $\Gamma$ is an arc of class $C^2$). Let $L$ be the length of $\Gamma$ and let $u = u(s)$, $v = v(s)$, where $0 \leqq s \leqq L$, be the parametric representation of $\Gamma$ with $s$ as the arc-length (in the metric (1)). Then $u(s)$, $v(s)$ are absolutely continuous and

$$g_{ik}(u(s), v(s))u_i'(s)u_k'(s) = 1, \qquad (u_1, u_2) = (u, v), \tag{6}$$

for all $s$ on $0 \leqq s \leqq L$, with the possible exception of an $s$-set of measure 0. Du Bois-Reymond's derivation of the Euler-Lagrange equations shows that, in virtue of (6), there exist constants $c_1$, $c_2$ for which

$$g_{jk}u_k' - \tfrac{1}{2}\int_0^s (\partial g_{ik}/\partial u_j)u_i'u_k'dt = c_j \qquad (j = 1, 2)$$

holds almost everywhere. Hence, if $(g^{ik})$ denotes the matrix inverse to $(g_{ik})$, then

$$u_m' = g^{jm}\left(\tfrac{1}{2}\int_0^s (\partial g_{ik}/\partial u_j)u_i'u_k'dt + c_j\right) \qquad (m = 1, 2)$$

holds almost everywhere. If $u_1'(s)$, $u_2'(s)$ are defined for *all* $s$ on $0 \leqq s \leqq L$ by the last formula line, then $u_1'(s)$, $u_2'(s)$ are continuous, while the functions $u_1(s) = \int_0^s u_1'(t)dt$, $u_2(s) = \int_0^s u_2'(t)dt$ coincide with the functions $u(s)$, $v(s)$ occurring in the above parametric representation of $\Gamma$.

Hence $\Gamma$ is of class $C^1$. Standard theorems in the calculus of variations imply that $\Gamma$ is of class $C^2$ and, therefore, a geodesic (in the sense of (2)). This completes the proof of (I).

**5. The first counter-example.** In the strip $|u| < 1$ of the $(u, v)$-plane, put

$$(7) \qquad g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = 1 - |u|^\lambda, \text{ where } 1 < \lambda < 2.$$

These functions $g_{ik}$ are of class $C^1$. The corresponding equations (2) become

$$(8) \qquad u'' + \tfrac{1}{2}\lambda(\operatorname{sgn} u)|u|^{\lambda-1}v'^2 = 0 \text{ and } ((1 - |u|^\lambda)v')' = 0.$$

The last equation gives

$$(9) \qquad (1 - |u|^\lambda)v' = c = \text{const.}$$

If $c = 0$, then $v = \text{Const.}$, while $u$ is a linear function of $t$; hence the parameter curves $v = \text{Const.}$ are geodesics. If $c \neq 0$, then $v' \neq 0$ along the geodesic, so that $v$ is strictly monotone along all geodesics except when $v = \text{Const.}$

If the independent variable is chosen to be the arc-length, then (6) is applicable. In view of (7), this means that $u'^2 + c^2(1 - |u|^\lambda)^{-1} = 1$, that is,

$$(10) \qquad u' = \pm(1 - c^2(1 - |u|^\lambda)^{-1})^{\frac{1}{2}}.$$

The case $c^2 = 1$ leads to the geodesic $u \equiv 0$.

Since every choice of $+$ or $-$ and every choice of $c$ on the interval $-1 \leqq c \leqq 1$ in (10) lead to a (locally) unique solution of (10), it now follows from (10) and (9) that *every direction through* $(u, v) = (0, 0)$ *determines a unique geodesic.*

**6.** It will now be shown that the example (7) proves all but the fifth of the six negations in (II).

*Ad* (i). Consider a geodesic with an initial direction near $(u', v') = (0, 1)$; this corresponds to a value of $c$ in (10) which is less than 1 but is close to 1. For convenience, it will be assumed that the $+$ sign is chosen in (10) for $u = 0$. Let $c = (1-\epsilon)^{\frac{1}{2}} > 0$, where $\epsilon > 0$, and let the corresponding geodesic be called $\Gamma_{+\epsilon}$. Then $u' > 0$ at $u = 0$, and $u'$ decreases until it becomes 0 at $u = \epsilon^{1/\lambda}$. Subsequently, $u$ decreases, since otherwise the right-hand side of (10) would become imaginary (so that the $-$ sign must then be chosen in (10)). If $v = v_0(\epsilon)$ denotes the value of $v$ at the point $u = \epsilon^{1/\lambda}$ of $\Gamma_{+\epsilon}$, then it is clear from the fact that $v$ does not occur explicitly in (7), that $\Gamma_{+\epsilon}$ is symmetric with respect to the line $v = v_0$. Hence the geodesic $\Gamma_{+\epsilon}$ joins the point $(0, 0)$ with $(0, 2v_0)$. Clearly, these two points are joined by the geodesic $u \equiv 0$ also.

It will now be shown that $v_0(\epsilon) \to 0$ as $\epsilon \to 0$. This will imply that, in any neighborhood $C_\epsilon$ of $(u, v) = (0, 0)$, there are points $(0, 2v_0)$ which can be joined with $(0, 0)$ by more than one geodesic contained in $C_\epsilon$. Hence, (7) shows that the answer to (i) is in the negative.

Let $s$ denote the arc-length (in the metric (7)) on $\Gamma_{+\epsilon}$ measured from $(u, v) = (0, 0)$. If $\epsilon < \frac{1}{2}$, it follows from (9), with $c = (1-\epsilon)^{\frac{1}{2}}$, that $0 < v' \leqq 2$; so that $v(s) \leqq 2s$ if $s > 0$. In particular, if $s = s_0 > 0$ is the (first) value of $s$ satisfying $u(s) = \epsilon^{1/\lambda}$, then $v_0 \leqq 2s_0$. Hence, in order to prove that $v_0(\epsilon) \to 0$, as $\epsilon \to 0$, it is sufficient to show that $s_0 = s_0(\epsilon) \to 0$.

Clearly, (10) implies that

$$s_0 = \int_0^a \{1 - (1-\epsilon)(1-u^\lambda)^{-1}\}^{-\frac{1}{2}} du, \text{ where } a = \epsilon^{1/\lambda}.$$

A simplification of the integrand shows that $s_0 = \int_0^a (1-u^\lambda)^{\frac{1}{2}}(\epsilon - u^\lambda)^{-\frac{1}{2}} du$,

hence, if $t = (\epsilon - u^\lambda)^{\frac{1}{2}}\epsilon^{-\frac{1}{2}}$ is introduced as a new integration variable,

$$\frac{1}{2}\lambda s_0 = \epsilon^{-\frac{1}{2}+1/\lambda} \int_0^1 (1 - \epsilon(1-t^2))^{\frac{1}{2}}(1-t^2)^{-1+1/\lambda} dt. \tag{11}$$

In view of $\lambda < 2$, this implies that $s_0 \to 0$ as $\epsilon \to 0$.

*Ad* (ii) It is clear that the arc $\Gamma_{-\epsilon}$, obtained by reflecting $\Gamma_{+\epsilon}$ across the line $u \equiv 0$, is also a geodesic (corresponding to the $-$ sign in (10) at

$s = 0$). Hence, if $v_0$ is positive and sufficiently small, there exist at least three geodesics joining $(0, 0)$ with $(0, 2v_0)$; namely, $u \equiv 0$ and both $\Gamma_{+\epsilon}$ and $\Gamma_{-\epsilon}$ for suitable $\epsilon > 0$. Actually, there is an infinity of geodesics joining $(0, 0)$ with $(0, 2v_0)$; for example, the "extension" of a geodesic joining $(0, 0)$ with $(0, 2v_0/n)$, where $n = 2, 3, \cdots$, also joins $(0, 0)$ with $(0, 2v_0)$.

Let $v_0 > 0$, and let $\Gamma$ be an arc joining $(0, 0)$ with $(0, 2v_0)$ and minimizing the arc-length in the sense of (ii). Then, if $v_0$ is sufficiently small, $\Gamma$ is a geodesic; cf. the proof of (I). It will be shown that $\Gamma$ is not a segment of $u \equiv 0$. Consequently, the arc obtained by a reflection of $\Gamma$ across the line $u \equiv 0$ is also a minimizing arc. Hence, the minimizing arc is not unique; so that (7) shows that the answer to (ii) is in the negative. (Actually, a refinement of the considerations to follow shows that the arcs $\Gamma_{+\epsilon}$ and $\Gamma_{-\epsilon}$, occurring above, are the only minimizing arcs joining $(0, 0)$ with $(0, 2v_0)$, when $v_0$ is sufficiently small.)

The length (in the metric (7)) of the line-segment joining $(0, 0)$ with $(0, 2v_0)$ is $2v_0$. It follows from (10) that the length of the corresponding sub-arc of $\Gamma_{+\epsilon}$ is $2s_0$. Thus it is sufficient to verify that $s_0 < v_0$ for small $\epsilon$. But (9), for $c = (1 - \epsilon)^{\frac{1}{2}}$, and (10) imply that

$$v_0 = (1 - \epsilon)^{\frac{1}{2}} \int_0^a (1 - u^\lambda)^{-\frac{1}{2}} (\epsilon - u^\lambda)^{-\frac{1}{2}} du, \text{ where } a = \epsilon^{1/\lambda}.$$

If $t = (\epsilon - u^\lambda)^{\frac{1}{2}} \epsilon^{-\frac{1}{2}}$ is introduced as a new integration variable, it follows that

$$\tfrac{1}{2}\lambda s_0 = \epsilon^{-\frac{1}{2}+1/\lambda}(1 - \epsilon)^{\frac{1}{2}} \int_0^1 (1 - \epsilon(1 - t^2))^{-\frac{1}{2}} (1 - t^2)^{-1+1/\lambda} dt.$$

This relation and (11), respectively, imply that if $\tfrac{1}{2}\lambda\epsilon^{\frac{1}{2}-1/\lambda} v_0$ and $\tfrac{1}{2}\lambda\epsilon^{\frac{1}{2}-1/\lambda} s_0$ are defined to be their common limiting value when $\epsilon \to 0$, then both of the resulting functions are differentiable at $\epsilon = 0$, having there the derivatives

$$-\tfrac{1}{4}\lambda \int_0^1 (1 - t^2)^{1/\lambda} dt \text{ and } -\tfrac{1}{2} \int_0^1 (1 - t^2)^{1/\lambda} dt,$$

respectively. Since $-\tfrac{1}{4}\lambda > -\tfrac{1}{2}$, it follows that the desired inequality, $v_0 > s_0$, is satisfied for sufficiently small $\epsilon > 0$.

*Ad* (iii). These considerations show that no minimizing arc (in the sense of (ii)) which joins $(0, 0)$ with a nearby point, $(u, v)$, has the direction of the $v$-axis at $(0, 0)$.

*Ad* (iv). Cf. the italicized statement at the end of § **5**.

(v) will be settled in § **7**.

*Ad* (vi). As found at the end of the treatment of (ii), there are three geodesics joining $(u, v) = (0, 0)$ with a nearby point $(0, v)$; namely, $\Gamma_{+\epsilon}$, $\Gamma_{-\epsilon}$ and $u \equiv 0$. But the geodesic $u \equiv 0$ did not furnish the minimal arc-length. Hence, in order to conclude that the answer to (vi) is in the negative, it is sufficient to ascertain that the initial point $(0, 0)$ can be replaced by any point of the geodesic $u \equiv 0$. But this is obvious, since, the functions (7) being independent of $v$, nothing is altered if the $(u, v)$-plane is arbitrarily translated in the direction of the $v$-axis.

*Remark.* In view of the (local) behavior of geodesics on a surface of class $C^2$ (cf. [5]), it is interesting to note that there exists a surface (of revolution, and admitting of a parametric representation of class $C^1$) for which (7) is the metric tensor. In fact, if $f(u) = (1 - |u|^\lambda)^{\frac{1}{2}}$, and if $g(u)$ is defined, for small $|u|$, by the quadrature assigned by $(df/du)^2 + (dg/du)^2 = 1$, then a surface of the type just mentioned results by revolving the arc $(y = f(u),\ z = g(u))$ about the $z$-axis (cf., e. g., [2], p. 67).

**7. The second counter-example.** In the $(u, v)$-plane, put

$$g_{11} = 1 + |v|^\lambda, \quad g_{12} = 0, \quad g_{22} = 1 + |v|^\lambda, \text{ where } 1 < \lambda < 2. \tag{12}$$

These functions are of class $C^1$. The corresponding equations (2) are

$$((1 + |v|^\lambda)u')' = 0 \text{ and} \tag{13}$$

$$v''(1 + |v|^\lambda) - \tfrac{1}{2}\lambda(\operatorname{sgn} v)|v|^{\lambda-1}(u'^2 + v'^2) = 0.$$

Only the geodesics issuing from $(u, v) = (0, 0)$ will be considered.

It is sufficient to deal with the half-plane $v \geqq 0$ (so that the sign of absolute value can be omitted in (13)). The first equation of (13) gives

$$(1 + v^\lambda)u' = c = \text{const.} \tag{14}$$

The choice $c = 0$ in (14) shows that $u \equiv 0$ is a geodesic. If $c \neq 0$, then $u' \neq 0$ along the geodesic, so that $u$ can be introduced as a parameter on all geodesics, except $u \equiv 0$, issuing from $(0, 0)$.

Since (12) reduces (6) to $(1 + v^\lambda)(v'^2 + c^2(1 + v^\lambda)^{-2}) = 1$, it follows from (14) that either $c = 0$ or

$$dv/du = \{k(1 + v^\lambda) - 1\}^{\frac{1}{2}}, \text{ where } k = 1/c^2. \tag{15}$$

If $k = 1$, then a solution of (15) satisfying $v(0) = 0$ is not unique. In fact, if $u_0 \geqq 0$ is arbitrary, then

$$v(u) = 0 \text{ if } 0 \leqq u \leqq u_0 \text{ and } v(u) = \{\tfrac{1}{2}(2-\lambda)(u-u_0)\}^{2/(2-\lambda)} \text{ if } u > u_0$$

represent a solution of (15) when $k = 1$.

*Ad* (v). In order to show that the answer to (v) is in the negative in the present case, it remains to show that every point $(u, v)$ can be joined with $(0, 0)$ by at most one geodesic arc. It will be sufficient to consider points $(u, v)$ in the quadrant $u \geqq 0, v \geqq 0$.

All geodesics, except $u \equiv 0$, are obtained from (15) by choosing $k \geqq 1$. Geodesics belonging to different values of $k$ $(\geqq 1)$ in (15) have no point $(u, v) \neq (0, 0)$ in common. In order to prove this, suppose the contrary. Then there exist two values of $k$, say $k_1$ and $k_2$ $(< k_1)$, such that the corresponding geodesics, say $v = v_1(u)$ and $v = v_2(u)$, intersect at some $(u^0, v^0) \neq (0, 0)$. Since $v_1'(0) > v_2'(0)$ by (15), it can be supposed that $v_1(u) > v_2(u)$ if $0 < u < u^0$. But then (15) also implies that $v_1'(u^0) > v_2'(u^0)$, since $v_1(u^0) = v_2(u^0) = v^0$. This is a contradiction.

If $k > 1$, then (15) has a (locally) unique solution $v = v(u)$ satisfying $v(0) = 0$. If $k = 1$, it is seen from the last formula line that the set of points common to two solution curves $v = v(u)$ of (15) which satisfy $v(0) = 0$ is either the single point $(u, v) = (0, 0)$ or a segment of the form $(0 \leqq u \leqq u_0, v = 0)$. Furthermore, any geodesic resulting from $k > 1$ has no point $(u, v) \neq (0, 0)$ on the geodesic $u \equiv 0$. Hence, any point $(u, v)$ can be joined with $(0, 0)$ by at most one geodesic.

**8. Proof of (III).** The assumptions of (III) imply that the Christoffel symbols $\Gamma^i_{jk} = \Gamma^i_{jk}(u, v)$ satisfy a uniform Lipschitz condition in a neighborhood of $(0, 0)$. Hence, any solution of (2) is uniquely determined by its initial conditions. For convenience, it can be supposed that the functions $g_{ik}$ are normalized as follows: $g_{11}(0,0) = g_{22}(0, 0) = 1$ and $g_{12}(0, 0) = 0$. Let (3) denote the geodesic $\Gamma_\phi$ determined by the initial conditions $u(0, \phi) = v(0, \phi) = 0$; $u_r(0, \phi) = \cos\phi$, $v_r(0, \phi) = \sin\phi$. Then $r$ is the arc-length (in the mertic (1)) on $\Gamma_\phi$. The functions (3) are defined and continuous on some circle $\mathfrak{C}$: $(0 \leqq r < a, 0 \leqq \phi < 2\pi)$.

The assumption that the function $g_{ik}(u, v)$ possess partial derivatives satisfying a uniform Lipschitz condition in a (simply connected) neighborhood of $(u, v) = (0, 0)$ has the following consequence: There exists a bounded measurable function $K = K(u, v)$, the "curvature" of (1), for which $K$

the formula of Gauss-Bonnet is valid. By this is meant that if $S$ is a simple closed curve which is piecewise of class $C^2$, and $T$ is the interior of $S$, then

$$2\pi - \sum_{j=1}^{n} \alpha_j - \int_S \kappa ds = \iint_T (g_{11}g_{22} - g_{12}^2)^{\frac{1}{2}} K du\, dv, \tag{17}$$

where $\alpha_1, \cdots, \alpha_n$ are the exterior angles (in the metric (1)) at the vertices of $S$, and $\kappa$ denotes the geodesic curvature along $S$. The idea of using (17) as a definition of the Gaussian curvature of (1) is due to Weyl [8], pp. 42-44.

The relation (17) can be verified as follows: Let $g_{ik}^n(u, v)$, where $n = 1, 2, \cdots$, denote three sequences of smooth functions which, on some neighborhood of $(u, v) = (0, 0)$, tend uniformly to the given $g_{ik}(u, v)$ in such a way that, as $n \to \infty$, the sequences of the partial derivatives $\partial g_{ik}^n/\partial u_j$ and $\partial^2 g_{ik}^n/\partial u_h \partial u_j$ are uniformly convergent and uniformly bounded, respectively. For fixed $S$ (and $T$), let $(17_n)$ denote the identity which results if the $g_{ik}$ are replaced by $g_{ik}^n$ in (17). Since the left-hand side of $(17_n)$ depends only on $S$, $\partial g_{ik}^n/\partial u_j$, $g_{ik}^n$ (and $\alpha_j = \alpha_j^n$), the left-hand side of $(17_n)$ tends to that of (17), as $n \to \infty$. Hence the right-hand side of $(17_n)$, say $\Phi_n(T)$, tends to a limit, say $\Phi(T)$, as $n \to \infty$. Since each set-function $\Phi_n(T)$ is a Lebesgue integral over $T$, and since the integrands are uniformly bounded (with respect to $u$, $v$ and $n$), it follows that the limit set-function $\Phi(T)$ is a Lebesgue integral, over $T$, of a bounded function. This proves (17).

**9.** The proof that $u_r(r, \phi)$, $v_r(r, \phi)$ are of class $C^1$ on a sufficiently small circle, can be obtained as a modification of the method used in [5]. If $|\phi|$ is sufficiently small, $u$ can be used as parameter on the arc $\Gamma_\phi$, for sufficiently small $u$. Let $v = v(u; \phi)$, where $0 \leqq u \leqq u_0$, denote this parameterization $\Gamma_\phi$; so that $v(0; \phi) = 0$ and $v_u(0; \phi) = \tan \phi$. As in [5], beginning of § 4, it is sufficient to show that the partial derivative $v_u(u; \phi)$ is of class $C^1$ in a wedge $(0 \leqq u \leqq u_0, |\phi| < \phi_0)$.

For given $\phi$, $\mu > 0$ and $\Delta\phi \neq 0$, let $S$ denote the closed curve consisting of the portion $0 \leqq u \leqq \mu$ of $\Gamma_\phi$; of the segment on the line $u = \mu$ which joins $(u, v) = (\mu, v(\mu; \phi))$ with $(u, v) = (\mu, v(\mu; \phi + \Delta\phi))$; finally, the portion $0 \leqq u \leqq \mu$ of $\Gamma_{\phi+\Delta\phi}$. Then $S$ is piecewise of class $C^2$, with possible corners only at the points $(0, 0)$, $(\mu, v(\mu, 0))$ and $(\mu, v(\mu, \phi + \Delta\phi))$. At this stage of the proof, it is not evident that the closed curve $S$ is a Jordan curve. Nevertheless, it is clear that the interior $T$ can be defined and (17) can be applied. The result is

$$\Delta\phi + \pi - \beta - \alpha - \int_{v_1(\mu)}^{v_2(u)} \kappa ds = \int_0^{\mu} \int_{v_1(\mu)}^{v_2(\mu)} (g_{11}g_{22} - g_{12}^2)^{\frac{1}{2}} K\, dv\, du, \tag{18}$$

where $\pi - \Delta\phi$, $\alpha$, $\beta$ are the suitably oriented angles (in the metric (1)) at the corners of $S$, and $v_1(u)$, $v_2(u)$ are abbreviations for $v(u;\phi)$ and $v(u;\phi+\Delta\phi)$, respectively, finally $\kappa$ denotes geodesic curvature along the line $u=\mu$.

Along the line $u=\mu$, consider a field of parallel vectors (parallel with reference to the metric (1)). If $\theta=\theta(v)$ is a continuous determination of the angle from a vector in this field to the line $u=\mu$, then the integral on the left of (18) is $\Delta\theta=\theta(v_2)-\theta(v_1)$. Also, if $w=w(v)=(w_1(v),w_2(v))\neq(0,0)$ denotes, on the line $u=\mu$, any continuous field of vectors for which $w(v_1)$, $w(v_2)$ are directed along the vectors tangent to $\Gamma_\phi$, $\Gamma_{\phi+\Delta\phi}$, respectively, and if $\omega=\omega(v)$ is a continuous determination of the angle from $w$ to the line $u=\mu$, then $(\pi-\beta)-\alpha=\Delta\omega=\omega(v_2)-\omega(v_1)$. Hence, the left-hand side of (18) is $\Delta\phi+\Delta\psi$, where $\psi=\omega-\theta$ is the angle from the vector $w$ to the corresponding vector in the given field of parallels.

If $w(v)$ is of class $C^1$, it is easily seen that $\psi=\psi(v)$ is of class $C^1$. Let

$$w_1(v)\equiv 1 \text{ and } w_2(v)=\{(v_2-v)v_{1u}+(v-v_1)v_{2u}\}/(v_2-v_1),$$

where $v_1=v_1(\mu)$, $v_2=v_2(\mu)$, $v_{1u}=v_{1u}(\mu)$ and $v_{2u}=v_{2u}(\mu)$ (the subscript $u$ denotes differentiation with respect to $u$). Then $d\psi/dv$ is

$$(g_{11}g_{22}-g_{12})^{\frac{1}{2}}g_{22}^{-\frac{1}{2}}(g_{ik}w_iw_k)^{-1} \text{ times } w_{2v}+\Gamma^1{}_{12}w_2+\Gamma^1{}_{22}w_2{}^2-\Gamma^2{}_{12}-\Gamma^2{}_{22}w_2,$$

where the argument of $g_{ik}$, $\Gamma^i{}_{jk}$ is $(\mu,v)$, and that of $w_2$, $w_{2v}$ is $v$ (cf., e. g., [7], pp. 131-132, where a similar formula is deduced under the assumption that $g_{ik}$ is the metric tensor of a surface.) Since $w_{2v}=(v_{2u}-v_{1u})/(v_2-v_1)=\Delta v_u/\Delta v$, and $w_2(v)=w_2(v_1)+o(1)$ as $\Delta\phi\to 0$, it follows that there exists a pair of continuous functions $p(\mu,v)>0$, $q(\mu,v)$ satisfying

$$\Delta\psi=(p(\mu,v_1)+o(1))\Delta v_u+(q(\mu,v_1)+o(1))\Delta v, \tag{19}$$

where $o(1)$ refers to $\Delta\phi\to 0$ and is uniform in $\mu$ for $0\leqq\mu\leqq u_0$. This derivation of (19) is valid under the tacit assumption that $v_1(\mu)\neq v_2(\mu)$. It is easily verified that (19) holds also if $v_1(\mu)=v_2(\mu)$.

For the sake of brevity, let $(g_{11}g_{22}-g_{12}{}^2)^{\frac{1}{2}}K$ be denoted by $M=M(u,v)$. Then the boundedness of $K$ implies that

$$\int_{v_1}^{v_2} M(u,v)\,dv=\{M(u,v_1(u))+\delta\}\Delta v(u),$$

where $\delta=\delta(u,\Delta\phi)\to 0$, as $\Delta\phi\to 0$, holds almost everywhere on $0\leqq u\leqq u_0$, and $\delta(u,\Delta\phi)$ is bounded. Hence, (18) can be written as

$$(20)\quad 1+(p+o(1))\Delta v_u/\Delta\phi+(q+o(1))\Delta v/\Delta\phi=\int_0^u (M+\delta)(\Delta v/\Delta\phi)du.$$

It follows from (20) that the functions $\Delta v(u;\phi)/\Delta\phi$ of $u$ are equicontinuous in $\phi$ and $\Delta\phi$, for $|\Delta\phi|\neq 0$ sufficiently small. Hence, by the usual arguments occurring in the theory of differential equations, $\Delta v(u;\phi)/\Delta\phi$ has a uniform limit, say $v_\phi(u;\phi)$, as $\Delta\phi\to 0$, and this limit is the unique function $y=y(u)$ satisfying $y(0)=0$ and the linear integro-differential equation

$$1+p(u,v(u;\phi))y'+q(u,v(u;\phi))y=\int_0^u M(\mu,v(\mu;\phi))y(\mu)d\mu.$$

This completes the proof that $u_r(r,\phi)$, $v_r(r,\phi)$ are of class $C^1$ on a sufficiently small circle $\mathfrak{C}$.

**10.** That (3) represents a topological mapping of the sufficiently small circle $\mathfrak{C}$ onto a neighborhood of $(u,v)=(0,0)$ and that the Jacobian $\partial(u,v)/\partial(r,\phi)\neq 0$ for $r\neq 0$ can be proved, exactly as in [5], end of § 4, by standard arguments involving Lipschitz's normal coordinates.

It only remains to verify that the element $ds^2$ of arc-length in the $(r,\phi)$-coordinates has the form (4). In what follows, the subscripts $r$ and $\phi$, and only these, will denote partial differentiations. Substitution of (3) into (1) shows that the coefficient of $dr^2$ in $ds^2$ is

$$(21)\qquad g_{ik}(u(r,\phi),v(r,\phi))u_{ir}u_{kr},\qquad (u_1,u_2)=(u,v).$$

But (21) must be identically 1, since $r$ is arc-length along each of the parameter curves $\Gamma_\phi$, where $\phi=\text{const}$. The coefficient of $2drd\phi$ in $ds^2$ is

$$(22)\qquad g_{ik}(u(r,\phi),v(r,\phi)u_{ir}u_{k\phi}.$$

Since $u_\phi=v_\phi=0$ holds identically in $\phi$ at $r=0$, the function (22) vanishes identically if it possesses a partial derivative with respect to $r$ and if this derivative is identically 0.

That (22) has a partial derivative with respect to $r$ follows from the fact that $u_r$ and $v_r$ are of class $C^1$. This partial derivative is the sum of

$$(23)\qquad (\partial g_{ik}/\partial u_j)(u_{jr}u_{ir}u_{k\phi})+g_{ik}u_{irr}u_{k\phi}$$

and

$$(24)\qquad g_{ik}u_{ir}u_{k\phi r}.$$

Since (21) is identically 1, it follows, by differentiating (21) with respect to $\phi$, that the expression (24) reduces to

$$-\tfrac{1}{2}(\partial g_{ik}/\partial u_j)u_{ir}u_{jr}u_{k\phi}. \tag{25}$$

Hence the partial derivative of (22) with respect to $r$ is the sum of (23) and (25). After the addition of (23) and (25), the coefficient of $u_{k\phi}$ is seen to become

$$g_{ik}u_i'' + (\partial g_{ik}/\partial u_j - \tfrac{1}{2}\partial g_{ij}/\partial u_k)u_i'w_j', \qquad (k=1,2), \tag{26}$$

where $' = \partial/\partial r$. It is easily verified that (26) can be written in the form

$$g_{ik}u_i'' + [ij,k]u_i'u_j', \qquad (k=1,2) \tag{27}$$

where $[ij,k]$ is a Christoffel symbol of the first kind. But the expressions (27) vanish by virtue of (2), since (3) represents a geodesic when $\phi = \text{const.}$ Consequently, the partial derivative of (22) with respect to $r$ is 0, and so (22) vanishes identically. This completes the proof of (III).

**11. Remark on a non-analytic torse of class $C^\infty$.** In the Appendix below, there will be proved a theorem dealing with the situation in which the $g_{ik}$ in (1) are just continuous. The assertion of the theorem will nevertheless be to the effect that, in a certain respect, the behavior of a minimizing arc is locally the same as in the Euclidean case. It is therefore somewhat unexpected that, even if the functions $g_{ik}$ are of class $C^\infty$, a minimizing arc $\Gamma$ can have the following property: If the Hilbert arc $\Gamma$ is thought of as an arc in the $(u,v)$-plane, then the line tangent to $\Gamma$ at a point $P$ of $\Gamma$ can intersect $\Gamma$ at an infinity of points which cluster at $P$.

In order to see this, let (1) be of the form

$$ds^2 = (1+f'^2)du^2 - 2f'du\,dv + dv^2,$$

where $f'$ denotes the derivative of a smooth function $f = f(u)$ of a single variable. Clearly, this $ds^2$ can be contracted into $ds^2 = du^2 + (dv - f'du)^2$. Accordingly, $ds^2 = dU^2 + dV^2$, if the parameters $U$, $V$ are defined by the transformation $U = u$, $V = v - f(u)$, the Jacobian of which is the constant $1 \neq 0$. Hence, the given $ds^2$ is isometric with the Euclidean $ds^2$. In particular, the line $V = 0$ in the $(U,V)$-plane is a geodesic, as well as a minimizing arc, $\Gamma$. But this $\Gamma$ is the curve $v = f(u)$ in the $(u,v)$-plane. Hence, in order to obtain a $\Gamma$ of the desired type, it is sufficient to choose $f(u) = \exp(-u^{-2})\sin(u^{-1})$ (if $u \neq 0$, and $f(0) = 0$).

## Appendix.

**12.** It was assumed in all of the above considerations that the coefficient function $g_{ik}(u, v)$ of the positive definite form (1) are of class $C'$, at least. On the other hand, it was pointed out by Hilbert himself ([6], p. 186) that his method leading to the extremal of the arc-length

$$|\int_P^Q ds| \tag{28}$$

applies also to (sufficiently small pieces of) surfaces $z = f(x, y)$ for which nothing but the existence of a continuous normal is assumed. Then, the function $f(x, y)$ being of class $C'$, the functions $g_{ik}$ occurring in (1) will be just continuous (perhaps nowhere differentiable), and therefore not in general of class $C'$. Actually, Hilbert's method supplies for (28) a minimizing arc also when all that is given (on a "geodesically convex" $(u, v)$-domain) is a set of three continuous functions $g_{ik}(u, v)$ for which the quadratic form (1) becomes positive definite, without being necessarily such as to be embeddable into an $(x, y, z)$-space.* In what follows, a smoothness property of the Hilbert minimizing curves will be proved which holds under this general assumption.

Let $\Gamma$ be a rectifiable Jordan arc in the given $(u, v)$-domain of the Gaussian parameters, and let $P, Q, \cdots$ be points on $\Gamma$. Consider the integral

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* It would be desirable to construct a binary symmetric matrix of the three continuous functions, $g_{ik}(u, v)$, corresponding to which the form (1) is positive definite but such as to fail to become realizable on any surface $X = X(u, v)$ of class $C'$ in the Euclidean space $X = (x, y, z)$. If the orders of continuous differentiability are raised by unity, that is, if the given functions $g_{ik}(u, v)$ are assumed to be of class $C'$ and the non-existence of a surface $X(u, v)$ of class $C''$ having the line element (1) has to be proved, then an example can be obtained as follows (cf. p. 554 of vol. 72 (1950) of this JOURNAL):

The functions $g_{ik}$ in (7) are of class $\bar{C}'$ for $-1 < u < 1$, $-\infty < v < \infty$ (and, as a matter of fact, their partial derivatives of first order satisfy a Hölder condition of index $\lambda - 1$), and have, except on the line $u = 0$, derivatives of arbitrarily high order. An application of the Theorema Egregium to the corresponding line-element (1) shows that, at every point $(u, v)$ at which $0 < |u| < 1$, the Gaussian curvature, $K$, is

$$K = -(2 - 2|u|^{\lambda})^{-2}\lambda^2 |u|^{2\lambda-2} + (2 - 2|u|^{\lambda})^{-1}\lambda(\lambda - 1)|u|^{\lambda-2},$$

where $1 < \lambda < 2$. Hence, $K = K(u, v)$ does not remain bounded as $(u, v) \to (0, 0)$. In particular, $K(u, v)$ cannot be defined at $(0, 0)$ so as to become continuous. Consequently, (1) cannot possess an embedding of class $C''$ into the $(x, y, z)$-space. For, if it did, the representation of $K$ as the quotient of the determinants of the second and first fundamental forms would supply the existence of a continuous $K(u, v)$ at $(u, v) = (0,0)$ also.

(28) along the arc of $\Gamma$ which is contained between $P$ and $Q$, with the understanding that the $ds$ in (28) is defined by (1), and let this integral (28) be divided by the integral (28) taken along the chord (that is to say, the rectilinear segment in the Euclidean $(u, v)$-plane) which joins $P$ with $Q$. Suppose that this ratio tends to 1 when $Q$ tends to $P$ on $\Gamma$ (while $P$ is fixed). Then $\Gamma$ will be said to possess the *Archimedean property* at $P$. Correspondingly, $\Gamma$ will be said to have the *uniform Archimedean property* if, as $Q \to P$, the ratio just mentioned tends to 1 uniformly in $P$, where $P$ is an arbitrary point of $\Gamma$.

In this terminology (cf. [1], Propositions 3-6, pp. 6-10), the result to be proved can be formulated as follows:

(*) *In a sufficiently small $(u, v)$-domain, let the three functions $g_{ik}(u, v)$ be continuous and such as to make the form* (1) *positive definite, and let $\Gamma$ be any, sufficiently small, minimizing arc (Hilbert) of* (28) *contained within the given $(u, v)$-domain. Then $\Gamma$ has the uniform Archimedean property.*

We were unable to decide whether or not, under the assumptions of (*) alone, the arc $\Gamma$, which always is a rectifiable Jordan arc, must have a tangent at each of its points (or, for that matter, whether or not $\Gamma$ must be an arc of class $C'$). Needless to say, the differential equations (2) cannot be written down under the present assumptions, since the Christoffel symbols $\Gamma^t_{jk}$ are not defined when the functions $g_{ik}$ are not differentiable.

The content of (*) can be illustrated by the following

COROLLARY 1. *Under the assumptions of* (*), *there cannot exist on $\Gamma$ a point which is a corner in the following sense: $\Gamma$ has right- and left-hand tangents without having a tangent.*

What is more, this property of lacking corners holds uniformly on $\Gamma$. In order to formulate this uniformity, consider the set $\Sigma = \Sigma_\eta$ of line-segments which join a point of the arc $PQ$ with a point of the straight line determined by $P$ and $Q$, and which form with that line an angle exceeding $\eta > 0$ and not exceeding $\pi - \eta > 0$. Then Corollary 1 of (*) can be refined as follows:

COROLLARY 2. *If the assumptions of* (*) *are satisfied and if $h = h_\eta$ denotes the least upper bound of the lengths of line-segments occurring in $\Sigma_\eta$, then there exists, for every $\epsilon > 0$, a $\delta = \delta_\epsilon$ having the property that*

$$h_\eta < \epsilon \mid PQ \mid \text{ whenever } \mid PQ \mid < \delta \tag{29}$$

*holds for any pair of points $P$, $Q$ on $\Gamma$.*

In (29), the sign of absolute value denotes distance (for instance, Euclidean distance in the $(u, v)$-plane).

In the proof of the above statements, the following notation will be used: $\mathrm{e}(PQ)$, $\mathrm{d}(PQ)$, $\mathrm{d}_P(PQ)$ will denote the *lengths of the arc* $PQ$ *of* $\Gamma$ in the respective metrics

$$du_i du_i, \quad g_{ik}(u_1, u_2)\,du_i du_k, \quad g_{ik}(u_1^0, u_2^0)\,du_i du_k,$$

the first being the Euclidean metric in the $(u, v)$-plane, the second the metric (1), and the third, in which $P = (u_1^0, u_2^0)$ is fixed, a metric which is affine-equivalent to the first. With references to these three metrics, $\mathrm{e}[PQ]$, $\mathrm{d}[PQ]$, $\mathrm{d}_P[PQ]$ will respectively denote the *lengths of the chord* $PQ$ *of* $\Gamma$.

**13.** *Proof of* (*). Let $C: u^2 + v^2 \leqq a^2$ be circle on which the functions $g_{ik}(u, v)$ are given as continuous and which contains $\Gamma$. Then, on the one hand, there exist positive constants $m$, $M$ satisfying (5) on $C$ and, on the other hand, there belongs to every $\epsilon > 0$ a $\delta = \delta_\epsilon$ having the property that if $P_1$, $P_2$ is any pair of points on $C$, then $|\, g_{ik}(P_1) - g_{ik}(P_2)\,| < \epsilon$ whenever $\mathrm{e}[P_1P_2] < \delta$. In view of Pythagorean inequality $(|\, a\,| + |\, b\,|)^{\frac{1}{2}} \leqq |\, a\,|^{\frac{1}{2}} + |\, b\,|^{\frac{1}{2}}$, this implies that, if $\xi_1$, $\xi_2$ are arbitrary as in (5), then

$$(30) \quad |\,(g_{ik}(P_1)\xi_i\xi_k)^{\frac{1}{2}} - (g_{ik}(P_2)\xi_i\xi_k)^{\frac{1}{2}}\,| \leqq \epsilon(\xi_1^2 + \xi_2^2)^{\frac{1}{2}}, \text{ where } \mathrm{e}[P_1P_2] < \delta.$$

Let $P$, $Q$ be two distinct point of $\Gamma$. Then it is clear from the definitions of the arc-lengths introduced at the end of Section 12, that

$$(31) \qquad |\,\mathrm{d}(PQ) - \mathrm{d}_P(PQ)\,| \leqq \epsilon\mathrm{e}(PQ),$$

provided that the arc $PQ$ of $\Gamma$ is within the circle $(u - u^0)^2 + (v - v^0)^2 < \delta^2$, where $P = (u^0, v^0)$. This proviso is satisfied if $\mathrm{d}(PQ) < M\delta$, where $M$ is the constant occurring in (5). Thus it follows from (31) that

$$(32) \qquad \mathrm{d}(PQ) \geqq \mathrm{d}_P(PQ) - \epsilon\mathrm{e}(PQ) \text{ if } \mathrm{d}(PQ) < M\delta.$$

On the other hand,

$$(33) \qquad \mathrm{d}_P(PQ) \geqq \mathrm{d}_P[PQ].$$

In fact, the distance $\mathrm{d}_P$ refers to the metric which is affine-equivalent to the Euclidean metric. Hence (33) follows from the fact that the geodesics of such a metric are straight lines.

Finally, (30) implies that, corresponding to (32),

$$\mathrm{d}_P[PQ] \geqq \mathrm{d}[PQ] - \epsilon \mathrm{e}[PQ] \text{ if } \mathrm{d}(PQ) < M\delta. \tag{34}$$

It follows from (32), (33) and (34) that

$$\mathrm{d}(PQ) \geqq \mathrm{d}[PQ] - \epsilon\{\mathrm{e}(PQ) + \mathrm{e}[PQ]\} \text{ if } \mathrm{d}(PQ) < M\delta. \tag{35}$$

On the other hand, $\mathrm{d}[PQ] \geqq \mathrm{d}(PQ)$, since $P$, $Q$ are on a Hilbert minimizing arc $\Gamma$. Hence, from (35),

$$1 \geqq \mathrm{d}(PQ)/\mathrm{d}[PQ] \geqq 1 - \epsilon\{\mathrm{e}(PQ) + \mathrm{e}[PQ]\}/\mathrm{d}[PQ] \text{ if } \mathrm{d}(PQ) < M\delta.$$

Consequently, from (5),

$$1 \geqq \mathrm{d}(PQ)/\mathrm{d}[PQ] \geqq 1 - 2\epsilon/m \text{ if } \mathrm{d}(PQ) < M\delta.$$

Since $m$, $M$ are independent of $\epsilon$ and $\delta = \delta_\epsilon$, the last formula line implies that $\mathrm{d}(PQ)/\mathrm{d}[PQ] \to 1$, as $Q \to P$, holds uniformly in $P$. This proves (*).

**14.** *Proof of Corollary* 1. Suppose, if possible, that the assertion of Corollary 1 is false. Then $\Gamma$ has some point $P$ at which there exist a right- and a left-hand tangent, say $l_1$ and $l_2$, but no tangent.

After an affine transformation of the $(u, v)$-plane, it can be assumed that the metric defining $\mathrm{d}_P$ is Euclidean. Let $\gamma$, where $0 < \gamma < \pi$, be the (Euclidean) angle between $l_1$ and $l_2$. With reference to the given $P$ and a sufficiently small $s > 0$, consider that arc $QR$ of $\Gamma$ which is bisected by $P$, in the following sense: $\mathrm{d}_P(QP) = \mathrm{d}_P(PR) = s$. If $\psi = \psi(s)$ denotes the (Euclidean) angle between the chords $QP$ and $PR$, then $\psi(s) \to \gamma$ as $s \to 0$. The perpendicular from $P$ to the chord $QR$ divides the angle $\psi$ into two angles $\alpha = \alpha(s)$, $\beta = \beta(s)$, having the sum $\psi$. Clearly,

$$\mathrm{d}_P[QR] = \mathrm{d}_P[QP] \sin\alpha + \mathrm{d}_P[PR] \sin\beta. \tag{36}$$

On the other hand, it is clear from the proof of (*) that

$$\mathrm{d}_P[QP] \sim \mathrm{d}_P[PR] \sim s \text{ as } s \to 0, \tag{37}$$

and similarly that

$$\mathrm{d}_P[QR] \sim \mathrm{d}[QR] \text{ as } s \to 0. \tag{38}$$

It is seen from (36) and (37) that $\mathrm{d}_P[QR] \sim s(\sin\alpha + \sin\beta)$. On the other hand, (38) and (*) imply that $\mathrm{d}_P[QR] \sim 2s$. Consequently, $\sin\alpha + \sin\beta \to 2$ as $s \to 0$. But this is impossible, since $\alpha + \beta \to \gamma$ as $s \to 0$, where $0 < \gamma < \pi$. This contradiction proves Corollary 1.

*Proof of Corollary* 2. In the proof of Corollary 1, only the Archimedean property of $\Gamma$ at a fixed point $P$ was used. If the full force of (*), claiming the uniform Archimedean property of $\Gamma$, is applied, the Corollary 2 follows in about the same way as Corollary 1 did.

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# ON THE ASYMPTOTIC CURVES OF A SURFACE.*

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The classical theory of asymptotic curves on surfaces of non-positive Gaussian curvature is based on assumptions of smoothness which are usually left unspecified. On the other hand, if the necessary assumptions are specified *ad hoc*, that is to say, so as to make directly applicable the standard existence and uniqueness theorems needed from the theory of differential equations, then there become involved hypotheses which not only appear to be too severe but at the same time such as to have nothing to do with the geometrical problems at hand.

The purpose of this paper is to dispose of this anomaly. On the one hand, conditions sufficient for the truth of the classical statements will be developed in terms which take into account the specifically geometrical origin of the various differential equations defining the respective problems. On the other hand, examples will show that the conditions to be imposed cannot be omitted.

Both an attraction and a difficulty of the theory is the circumstance that, even if the surface has a parametrization in terms of asymptotic curves, examples show that this parametrization of the surface *can* be less differentiable than some other parametrization of the surface (or, equivalently, less smooth than the parametrization $z = z(x, y)$, where $x, y, z$ are Cartesian coordinates).

Surfaces of negative Gaussian curvature are dealt with in Parts I, II, III. Part IV deals with lines of curvature and applies therefore to surfaces of positive curvature also. Finally, Part V develops the corresponding theory for torses.

## PART I. Problems of Uniqueness.

**1. Statement of the theorems.** Let $S$ be a small piece of a surface of class $C''$ in the $(x, y, z)$-space. Thus, if the Gaussian parameters $(u^1, u^2)$ on $S$ are suitably chosen, there exists on $S$ a first fundamental form, $g_{ik}du^i du^k$, which is positive definite, with coefficient functions $g_{ik} = g_{ik}(u^1, u^2)$ which are of class $C'$, and there exists a second fundamental form, $h_{ik}du^i du^k$, in which the coefficient functions $h_{ik} = h_{ik}(u^1, u^2)$ are continuous. The signa-

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ture of the second fundamental form depends on the Gaussian curvature $K = K(u^1, u^2)$, since $K = \det(h_{ik})/\det(g_{ik})$, where $\det(g_{ik}) > 0$.

Suppose that $K < 0$ on $S$. This means that the second fundamental form is indefinite throughout. Hence every point $(u^1, u^2)$ determines a distinct pair of directions, $du^1 : du^2$, characterized by the vanishing of the second fundamental form. This pair of directions is called the pair of asymptotic directions (at a given point of $S$). By an asymptotic curve on $S$ is meant a curve, of class $C'$, along which the direction of the tangent is always an asymptotic direction.

The papers and books known to us either do not specify a precise degree of smoothness required of the surface $S$, or else they imply that all asymptotic curves together must consist of two locally schlicht families of curves, which can therefore be introduced as parameter curves (if $S$ is small enough). A closer inspection of the questions shows, however, that the situation is much more involved.

The differential equation of the asymptotic curves is

$$h_{ik}du^i du^k = 0, \qquad (i = 1, 2; k = 1, 2), \tag{1}$$

where, by assumption, the functions $h_{ik} = h_{ik}(u^1, u^2)$ have a negative determinant. Hence, (1) can be split into two real differential equations of the form

$$dv/du = f(u, v), \tag{2}$$

where $u, v$ denote $u^1, u^2$ or $u^2, u^1$, respectively (the respective functions $f$ are not identical). But if $S$ is just of class $C''$ (and, correspondingly, the parameters, $u$ and $v$, on $S$ are so chosen that they supply a $C''$-parametrization of $S$), then the functions $h_{ik}(u, v)$ are just continuous, and so nothing like a Lipschitz condition is assured for the continuous function $f(u, v)$ occurring in (2). Hence, it cannot be concluded that (2) has only one solution $v = v(u)$ satisfying an initial condition, say $(v)_{u=0} = 0$.

There arises, however, the question whether the resulting possibility of non-uniqueness can actually occur. It will be proved below that the answer to this question is affirmative:

(i) *If a surface, $S$, is of class $C''$ and if the Gaussian curvature is negative at a point, $P$, of $S$, then $P$ can issue a continuum of asymptotic curves (each of which is tangent to one of the two distinct asymptotic directions at $P$).*

This is the more interesting as, under the standard assumptions of differentiability, the asymptotic curves are known to be identical with the

characteristics of the partial differential equations defining the problem of deformation (cf., e. g., [2], p. 279), as well as with the characteristics of the partial differential equations on which the classical problem of embedding depends (ibid., p. 252).

The assertion of (i) is anything but obvious. In fact, if $f(u, v)$ is given as an *arbitrary* continuous function corresponding to which the initial condition $(v)_{u=0} = 0$ determines more than one solution $v = v(u)$ of (2), then (2) cannot be "embedded" so as to become the differential equation of a family of asymptotic lines (on *some* $S$). Correspondingly, the proof of (i) will depend on a delicate construction, leading to an $f(u, v)$ for which that "embedding" is possible.

*Remark.* The claim of (i) becomes false if $C''$ is replaced by $C'''$ in (i). In fact, the functions $h_{ik}$ then become of class $C'$; hence, the $f$ in (2) is of class $C'$, and so Lipschitz's uniqueness condition is satisfied.

The negative statement of (i) makes clear the content of the following theorem:

(ii) *If a surface, $S$, of class $C''$ is of constant negative curvature, then either of the asymptotic directions at a point, $P$, of $S$ determines a unique asymptotic curve through $P$.*

The assertion of (ii) is not made illusory by the Remark preceding (ii). In fact, if $S$ is of class $C''$ and is given in the form $z = z(x, y)$, where $z$ is a function of class $C''$, then $K = \det(h_{ik})/\det(g_{ik})$ reduces to

$$K = (rt - s^2)/(1 + p^2 + q^2)^2, \tag{3}$$

where, as usual, $p = z_x, \cdots, t = z_{yy}$. Since the assumption of (ii) means that $K = K(x, y)$ is a negative constant, say $-1$, it follows that the surfaces dealt with in (ii) can be thought of represented by those functions $z = z(x, y)$ of class $C''$ which satisfy the partial differential equation

$$rt - s^2 + (p^2 + q^2 + 1)^2 = 0. \tag{4}$$

But the Monge-Ampère equation (4) (with constant, hence analytic, coefficients) is of hyperbolic type and will therefore undoubtedly have solutions $z = z(x, y)$ which are of class $C''$ but not smoother.

The proof of (ii) will be such as to allow a generalization to certain surfaces, to be called of class $C^{**}$, which are not of constant curvature. Let $S$ be called of class $C^{**}$ if it is of class $C''$ and if it has a $C''$-parametrization in which the Gaussian curvature satisfies a Lipschitz condition (as a function

of the parameters in question). It is clear that, if $S$ is of class $C^{**}$, the Gaussian curvature satisfies a Lipschitz condition in every $C''$-parametrization.

Clearly, a surface of class $C''$ which is of constant curvature must be of class $C^{**}$. Hence, (ii) is contained in the following theorem:

(iii) *If a surface, $S$, is of class $C^{**}$ and if the Gaussian curvature is negative at a point, $P$, of $S$, then either of the asymptotic directions at $P$ determines a unique asymptotic curve through $P$.*

Inasmuch as the proof of (i) is somewhat length, (iii) will be proved first.

**2. Proof of (iii).** There is no loss of generality in assuming that $S$ is given in the form $z = z(x, y)$, where $z(x, y)$ is a function of class $C''$ on the circle $x^2 + y^2 < 1$, and that the point $P$ in (iii) is $(x, y) = (0, 0)$. Then, since $K = K(x, y)$ is supposed to be negative at $(0, 0)$, it is seen from (3) that, after a suitable rotation of the $(x, y)$-plane, it can be assumed that

$$t(0, 0) \neq 0 \tag{5}$$

$(t = z_{yy})$. On the other hand, since $(u^1, u^2) = (x, y)$, the definition of the second fundamental form shows that (1) reduces to

$$r\,dx^2 + 2s\,dx\,dy + t\,dy^2 = 0. \tag{6}$$

It follows from (3) that (6) can be written in the form

$$s\,dx + t\,dy = \pm(-K)^{\frac{1}{2}}(1 + p^2 + q^2)\,dx. \tag{7}$$

It is understood that $s$, $t$, $p$, $q$, $K$ in (7) are given functions of $(x, y)$.

In a sufficiently small neighborhood of $(x, y) = (0, 0)$, replace the independent variables $x, y$ by

$$x = x, \qquad q = z_y(x, y) \tag{8}$$

(Legendre). Since $z(x, y)$ is of class $C''$, the (local) binary mapping $(x, y) \to (x, q)$ is of class $C'$ and (locally) schlicht. In fact, the Jacobian, $\partial(x, q)/\partial(x, y)$ of (8) is $z_{yy}$, which, in view of (5), is distinct from 0 at the origin and therefore, by continuity, near the origin.

Accordingly, (7) can be written in the form

$$dq = \pm(-K)^{\frac{1}{2}}(1 + p^2 + q^2)\,dx, \tag{9}$$

where $K$ and $p$ are considered as functions of the independent variables $x, q$. Hence, if it is ascertained that both of these functions, $K(x; q)$ and $p(x; q)$,

satisfy a Lipschitz condition with respect to $q$, then the proof of (iii) will be complete. In fact, either of the differential equations (9) will then be of the form (2), with $u = x$, $v = q$ and with an $f$ satisfying Lipschitz's condition.

Since the surface $S$ is supposed to be of class $C^{**}$, both $K = K(x, y)$ and $p = z_x(x, y)$ satisfy a Lipschitz condition in $x$ and $y$ together. On the other hand, the transformation (8) and its inverse are of class $C'$ and preserve, therefore, the existence of a Lipschitz constant (near the origin). Hence, the proof of (iii) is complete.

**3. Proof of (i).** Let $g(x)$, $h(y)$ denote continuous functions, each a function of a single variable, which satisfy

$$g(0) = 0, \qquad h(0) = 0, \tag{10}$$

and let $G(x)$, $H(y)$ then be defined by the quadratures assigned by $d^2G/dx^2 = g(x)$, $d^2H/dy^2 = h(y)$ and by arbitrary initial conditions. Then

$$z \equiv z(x, y) = G(x) + H(y) + xy \tag{11}$$

represents a surface of class $C''$. On this surface, the differential equation, (6), of the asymptotic curves reduces to

$$g(x)dx^2 + 2dxdy + h(y)dy^2 = 0. \tag{12}$$

Hence, it is seen from (10) that the surface (11) has a negative curvature at the point $(x, y) = (0, 0)$, and that the asymptotic directions at this point are those of the coordinate axes. But on an asymptotic curve issuing from $(0, 0)$ in the direction of the $x$-axis, the variable $x$ can be used as a parameter (for small $|x|$). Then (12) becomes

$$g(x) + 2y' + h(y)y'^2 = 0, \tag{13}$$

where the prime denotes differentiation with respect to $x$. Hence, (i) is a consequence of the following assertion:

(†) *On $x$- and $y$-intervals about $x = 0$ and $y = 0$, respectively, it is possible to define continuous functions, $g(x)$ and $h(y)$, satisfying* (10) *and having the following property: The differential equation* (13) *has more than one solution path $y = y(x)$ passing through the point $(x, y) = (0, 0)$.*

Needless to say, the $h(y)$ *in* (†) cannot satisfy a Lipschitz condition.

*Remark.* The quadratic equation (13) for $y'$ splits into two differential equations, linear in $y'$, one of which is $y' = f(x, y)$, where

$$f = \{-1 + (1 - gh)^{\frac{1}{2}}\}/h \text{ if } h \neq 0, \text{ and } f = -\tfrac{1}{2}g \text{ if } h = 0.$$

This $f(x,y)$ is continuous (for small $x^2+y^2$), and it is clear from the connection between (iii) and (†) that it is the above $f(x,y)$ corresponding to which the point $(x,y)=(0,0)$ fails to be a point of uniqueness of the differential equation $y'=f(x,y)$. But the latter clearly is the differential equation of the characteristics of a linear partial differential equation of second order,

$$z_{xx}+2f(x,y)z_{xy}+f^2(x,y)z_{yy}=0. \tag{14}$$

Since $1\cdot f^2-f\cdot f$ vanishes identically, (14) is of parabolic type. If $z=z(x,y)$ denotes the surface $S$ supplied by (iii), it is seen that (14) has a solution $z=z(x,y)$ of class $C''$ on an $(x,y)$-domain containing the point $(0,0)$. But the latter issues a continuum of characteristics (in the same direction), namely, those solutions of $y'=f(x,y)$ which satisfy $y(0)=0$. Hence, the classical methods, based on fields of characteristics, cannot lead to any solution $z=z(x,y)$ of (14).

**4. Proof of (†).** Let $w=w(x)$ be defined by

$$w(x)=-x^6+x^7\sin(x^{-5}) \text{ if } x\neq 0, \text{ and } w(0)=0. \tag{15}$$

Then $w(x)$ is of class $C'$; in fact, the derivative of (15) is given by

$$w'(x)=-6x^5-5x\cos(x^{-5})+7x^6\sin(x^{-5}) \text{ if } x\neq 0, \text{ and } w'(0)=0. \tag{16}$$

The function $y=w(x)$ satisfies $y(0)=0$, and is a solution of (13) if

$$g(x)=-2w'(x)-h(w(x))w'^2(x). \tag{17}$$

Let the continuous function $g(x)$ be defined by (17) in terms of $w(x)$ and of the continuous function $h(y)$ (the latter is unspecified as yet). The first of the conditions (10) is satisfied by virtue of (16) and (17).

It remains to show that the continuous function $h(y)$ can be so chosen that $h(0)=0$ and that (13) possesses a solution $y=y(x)$ satisfying $y(0)=0$ and $y(x)\not\equiv w(x)$, for small $|x|$. But such an $h$ results by placing

$$h(y)=|2y|^{1/3} \text{ if } y\leqq 0, \text{ and } h(y)=0 \text{ if } y>0. \tag{18}$$

In order to see this, define $y(x)$ by the quadratures assigned by

$$y'(x)=-\tfrac{1}{2}g(x) \text{ if } x\geqq 0 \text{ and } y'(x)=w'(x) \text{ if } x<0 \tag{19}$$

and by the initial condition $y(0)=0$. According to (17), this means that

$$(20)\quad y(x) = w(x) + \int_0^x \tfrac{1}{2}w'^2(t)h(w(t))\,dt \text{ if } x \geqq 0,$$
$$\text{and } y(x) = w(x) \text{ if } x < 0,$$

where $h$ is defined by (18). It will be shown that $y(x)$ is a solution of (13) for small $|x|$, and that $y(x) \not\equiv w(x)$ for small $x > 0$.

The function (15) satisfies $w(x) < 0$ for small $|x| \neq 0$. Hence, it will follow that $y(x) \not\equiv w(x)$ if it is shown that

$$(21)\qquad y(x) > 0 \text{ for small positive } x.$$

On the other hand, since $h(y) = 0$ for $y > 0$, it follows from (13) and the first part of (19), that (20) is a solution of (13) for small $x > 0$ if (21) holds. Thus, in order to complete the proof of (†), it is sufficient to verify (21).

**5. Proof of (21).** The definition (14) of $w(x)$ shows that $w(x) < -\frac{1}{2}x^6$ for small $|x| \neq 0$. Thus the monotony of the function $h(y)$ implies that $h(w(x)) > h(-\frac{1}{2}x^6) = x^2$. Hence, $y(x) \geqq w(x) + v(x)$ for small $x > 0$, if $v(x)$ is defined by

$$(22)\qquad v(x) = \int_0^x \tfrac{1}{2}t^2w'^2(t)\,dt.$$

It will be shown that, for some constant $C > 0$,

$$(23)\qquad v(x) > Cx^5 \text{ for small positive } x.$$

The existence of such a $C$ leads, for small positive $x$, to the inequalities

$$y(x) \geqq w(x) + v(x) > w(x) + Cx^5 > 0.$$

The last inequality follows from $C > 0$ and from (15). Thus, if the existence of a $C$ satisfying (23) is proved, the proof of (21) is complete.

In order to assure the existence of a $C$, note that (15) shows that, for some constant $c > 0$,

$$(24)\qquad x^2w'^2(x) \geqq cx^4,$$

provided that $\cos^2(x^{-5}) \geqq \frac{1}{2}$ and that $|x|$ is sufficiently small. These provisos are satisfied if

$$(25)\qquad a_k \leqq x \leqq b_k, \text{ where } a_k^{-5} = (k + \tfrac{1}{4})\pi \text{ and } b_k^{-5} = (k - \tfrac{1}{4})\pi,$$

when $k$ is a large positive integer. If $x > 0$ is sufficiently small and the

integer $k = k(x)$ is so chosen that $b_k \leqq x < a_{k-1}$, then $k^{-1} \sim \pi x^5$ as $x \to +0$. Furthermore, (22) implies that

$$v(x) > \tfrac{1}{2}c \sum_{j=k}^{\infty} \int_{a_j}^{b_j} t^4 dt = c \sum_{j=k}^{\infty} (b_j^5 - a_j^5)/10.$$

But the definitions of $a_j$ and $b_j$ show that

$$\sum_{j=k}^{\infty} (b_j^5 - a_j^5) > \text{const.} \sum_{j=k}^{\infty} j^{-2} > \text{Const.}\ k^{-1},$$

where const. $> 0$ and Const. $> 0$. Since $k^{-1} \sim \pi x^5$ as $x \to +0$, the last two formula lines prove the existence of a constant $C > 0$ satisfying (23).

## PART II. Asymptotic Parametrizations.

**6. The theorems.** According to (iii), every point $P$ of a surface $S$ of class $C^{**}$ of negative Gaussian negative curvature issues exactly two asymptotic curves. Consequently, there exist (locally schlicht) parametrizations of $S$ of the form $X = X(u, v)$, where $X = (x, y, z)$ is a *continuous* vector function of $(u, v)$, and $u = \text{const.}$ and $v = \text{Const.}$ are asymptotic curves. With a slight modification of the conditions of (iii), this assertion can be improved:

(iv) *Let $S$ be a surface of negative curvature, $K$, possessing a $C''$-parametrization in which $K$ is a function of class $C'$ (so that, in particular* (iii) *is applicable). Then, near any point of $S$, there exist parametrizations $X = X(u, v)$, where $X = (x, y, z)$ is a vector function of class $C'$ in $(u, v)$, and $u = \text{const.}$ and $v = \text{Const.}$ are asymptotic curves.*

In particular, if $S$ is a surface of class $C''$ and of constant negative curvature, then it possesses asymptotic parametrizations of class $C'$. This assertion seems to be curious, since, starting with a pseudo-sphere of class $C''$, it supplies a parametrization which is of class $C'$ only. Actually, it turns out that (iv) cannot be improved in this respect:

(v) *Under the assumptions of* (iv), *let $S$: $X = (x, y, z)$ be (locally) parametrized in the form $X = X(u, v)$, where $u = \text{const.}$ and $v = \text{Const.}$ are asymptotic curves. Then no such parametrization of $S$ need be of class $C''$.*

The assertion (v) remains true even if $S$ is of constant negative curvature. In fact, it will be shown (cf. (viii) below) that if $S$ is of constant negative curvature and if its (Cartesian) parametrization, say $z = z(x, y)$, is not of class $C'''$ near a point, then it *cannot* possess an asymptotic parametrization

of class $C''$ near that point. Thus, (v) is contained in (viii), and so a direct proof of (v) will not be given.

**7. Proof of (iv).** It can be assumed that $S$ is given in the form $z = z(x, y)$. Then it is clear from the assumptions of (iv) that the function $z(x, y)$ is of class $C''$ and that $K(x, y)$ is of class $C'$. Consequently, if (5), (8), (9) are used in the same way as in the proof of (iii), it is seen that the differential equations defining the asymptotic curves reduce to two equations of the form (2), with functions $f(u, v)$ of class $C'$. Since this implies that the (local) general solution, $v = v(u; u_0, v_0)$, of (2) is a function of class $C'$ in $u$, $u_0$ and $v_0 = v(u_0; u_0, v_0)$ together, the assertion of (iv) is now clear from the end of the proof of (iii).

## PART III. Asymptotic Tchebychef Nets.

**8. The existence theorem.** A parametrization $X = X(u, v)$ of a surface $S$ of class $C'$ will be called a Tchebychef parametrization (cf., e. g., [1], § 60-§ 63) if it is of class $C'$ and such as to lead to the following form of the squared line-element on $S$:

$$(26) \qquad |\, dX(u, v)\,|^2 = du^2 + 2\cos\phi\; du\, dv + dv^2, \text{ where } \phi = \phi(u, v).$$

If the subscripts of $X$ refer to partial differentiations, (26) is equivalent to

$$(27) \qquad |\, X_u(u, v)\,| \equiv 1 \text{ and } |\, X_v(u, v)\,| \equiv 1, \qquad (X_u \cdot X_v = \cos\phi).$$

Clearly, $\phi(u, v)$ is a continuous function.

It is a central fact in the differential geometry of pseudo-spheres that the latter possess asymptotic parametrizations satisfying (26) (for instance, Hilbert's theorem on the existence of singularities on pseudo-spheres depends on this fact). A careful perusal of the classical proof reveals however, that the surface is tacitly assumed to be of class $C''''$. On the other hand, it will turn out that, starting with (iv) alone, it is possible to effect a substantial reduction, $C'''' \to C''$, of the assumption of smoothness:

(vi) *Let a surface $S$ be of constant negative curvature ($K \equiv -1$) and of class $C''$. Then there exists, near every point of $S$, a $C'$-parametrization, $X = X(u, v)$, which is an asymptotic parametrization satisfying* (27), *i. e.*, (26).

**9. Proof of (vi).** By the remarks following (iv), the assumptions of (iv) imply, for a vicinity of every point of $S$, the existence of an asymptotic

$C'$-parametrization. Let $X = X(u, v)$ be such a parametrization. Then $u = \text{const.}$ and $v = \text{Const.}$ are asymptotic curves, and the first partial derivatives, $X_u$ and $X_v$, of the vector function $X$ exist and are continuous. It will be shown that this is possible only if the second mixed partial derivative

(28) $$X_{uv} = X_{vu} \text{ exists and is continuous}$$

and represents a vector parallel to the vector product $[X_u, X_v]$:

(29) $$X_{uv} \cdot X_u = 0, \qquad X_{uv} \cdot X_v = 0.$$

If (28) and (29) are granted at every point $(u, v)$ of $S$, the proof of (vi) is straightforward. In fact, (28) and (29) imply that $(|X_u|^2)_v = 0$ and $(|X_v|^2)_u = 0$ hold identically. This means that $|X_u|^2$ is a function of $u$ alone, and $|X_v|^2$ a function of $v$ alone, say $|X_u(u, v)| = f(u)$ and $|X_v(u, v)| = g(v)$. Finally, the transition from the latter relations to the assertion, (27), of (vi) requires only a $C'$-transformation, $(u, v) \to (u^*, v^*)$, of the trivial form $u^* = u^*(u)$, $v^* = v^*(v)$, that is, a re-parametrization along the asymptotic curves.

Accordingly, only (28) and (29) remain to be proved. Actually, (28) holds if the assumptions of (vi) are generalized to those of (iv):

(vi bis) *If $S$ satisfies the assumptions of* (iv), *then* (28) *holds for every asymptotic $C'$-parametrization supplied by* (iv).

If (vi bis) is proved, (vi) will follow if (29) is verified under the additional assumption, $K \equiv -1$, of (vi).

**10. Proof of (vi bis).** If

(30) $$l = (-K)^{-\frac{1}{2}}(1 + p^2 + q^2)^{-1},$$

then, by (9),

(31) $$x_u = lq_u, \qquad x_v = -lq_v,$$

where $X = X(u, v)$ is the asymptotic $C'$-parametrization supplied by (iv) and $X = (x, y, z)$. The choice, $+$, made in (30) and (31) for the alternative sign, $\pm$, in (9) is admissible, since $u$ and $v$ can be interchanged.

Clearly, $x$, $q$ in (9) can be replaced by $y$, $p$, respectively. Then what corresponds to (31) is

(32) $$y_u = -lp_u, \qquad y_v = lp_v,$$

the choice of the alternative sign, $\pm$, being committed by (31). In fact, in

order to see the effect of the commitment in (31), it is sufficient to write (6) in the form

$$dp\,dx + dq\,dy = 0, \tag{33}$$

and to substitute (32) into (33).

In the (euclidean) plane of the parameters $u$, $v$, let $R$ be the positively oriented boundary of a small rectangle, having the vertices $(u_1, v_1)$, $(u_2, v_1)$, $(u_2, v_2)$, $(u_1, v_2)$, where $u_1 < u_2$, $v_1 < v_2$, and put

$$[x] = x(u_1, v_1) - x(u_2, v_1) + x(u_2, v_2) - x(u_1, v_2). \tag{34}$$

Then it is seen from (31) that

$$[x] = -\tfrac{1}{2}\int_R (lq_u\,du + lq_v\,dv). \tag{35}$$

Consider the point transformation $(u, v) \to (p, q)$ defined by the inverse of

$$p = z_x(x, y), \quad q = z_y(x, y), \text{ where } x = x(u, v), \quad y = y(u, v). \tag{36}$$

Since $z$ is a function of class $C''$ in $x, y$, and since the components of $X(u, v)$ are of class $C'$ in $u, v$, it is clear that the transformation $(u, v) \to (p, q)$ is of class $C'$. It has a local inverse of class $C'$, since its Jacobian,

$$J = \partial(p, q)/\partial(u, v), \tag{37}$$

does not vanish. In fact, (37) is identical with the product of $rt - s^2$ and $\partial(x, y)/\partial(u, v)$, where $rt - s^2 < 0$, by (3), and $\partial(x, y)/\partial(u, v) \neq 0$, since $X = X(u, v)$ is an admissible $C'$-parametrization. Incidentally, there is no loss of generality in assuming that (37) is positive; for, if it is negative, it becomes positive upon the substitution $(u, v) \to (-u, v)$. But the latter can be admitted, since it amounts to a re-orientation of the asymptotic curves $v = \text{Const.}$

Accordingly, if $R$ is small enough, the substitution $(u, v) \to (p, q)$ is applicable to (35). This gives

$$[x] = -\tfrac{1}{2}\int_R l\,dq; \text{ hence, } [x] = -\tfrac{1}{2}\iint_Q l_p\,dp\,dq,$$

if $Q$ denotes the interior of $R$ (Green). Consequently, if $p$, $q$ are replaced by $u$, $v$ in the last integral, it follows that

$$[x] = -\tfrac{1}{2}\int_{v_1}^{v_2}\int_{u_1}^{u_2} l_p J \, du \, dv, \tag{38}$$

the Jacobian (37) being positive.

If $R$ is made to shrink to a point $(u, v)$, it is seen from (34) and (38) that the second mixed partial derivative $x_{uv}$ at that point $(u, v)$ exists, is equal to $x_{vu}$ and is represented by

$$x_{uv} = -\tfrac{1}{2} l_p J. \tag{39}$$

In view of (30), this implies that $x_{uv} = x_{vu}$ is continuous. It is proved in the same way the $y_{uv}$ and $y_{vu}$ exist, are equal and continuous and that, corresponding to (39),

$$y_{uv} = -\tfrac{1}{2} l_q J. \tag{40}$$

Hence, in order to complete the proof of (vi bis), only the existence of a continuous $z_{uv}$ remains to be ascertained. But this follows by differentiating, with respect to $v$, the identity

$$z_u = p x_u + q y_u, \tag{41}$$

in which each of the functions $p$, $q$, $x_u$, $y_u$ possesses a continuous partial derivative with respect to $v$.

**11. Completion of the proof of (vi).** What remains is the verification of (29) under the assumption $K \equiv -1$. But the latter reduces (30) to

$$l = (1 + p^2 + q^2)^{-1}. \tag{42}$$

It follows therefore from (39) and (40) that

$$x_{uv} = p l^2 J, \qquad y_{uv} = q l^2 J. \tag{43}$$

In addition,

$$z_{uv} = -l^2 J. \tag{44}$$

In fact, differentiation of (41) with respect to $v$ gives

$$z_{uv} = p x_{uv} + q y_{uv} + p_v x_u + q_v y_u.$$

Hence, if $x_{uv}$, $y_{uv}$ are substituted from (43), and $x_u$, $y_u$ from (31), (32),

$$z_{uv} = (p^2 + q^2) l^2 J + (p_v q_u - p_u q_v) l.$$

But this can be contracted into the form (44) if use is made of the definitions, (37) and (42), of $J$ and $l$.

Let $N = N(x, y)$ denote the normal vector of $S: z = z(x, y)$, oriented in such a way that the direction cosines of $N$ become a scalar multiple of the vector $(p, q, -1)$. Then it is seen from (43) and (44) that, if $S$ is represented in the form $X = X(u, v)$, the vector $X_{uv} = (x_{uv}, y_{uv}, z_{uv})$ is a scalar multiple of $N$. Since the tangent plane of $S$ is spanned out by the vectors $X_u$, $X_v$ and is normal to $N$, it follows that the vector $X_{uv}$ is perpendicular to both vectors $X_u$, $X_v$, as claimed by (29).

This completes the proof of all the statements made above.

**12. On the smoothness of $\varphi(u, v)$ in (26).** The assertion (vi) corresponds to (iv); what corresponds to (v) is the statements that the continuous function $\phi(u, v)$ in (26) need not be of class $C'$. Actually, it will turn out that $\phi(u, v)$ cannot be of class $C'$ unless $S$ is of class $C'''$ (in some parametrization, say $z = z(x, y)$). In this direction, the following fact will first be proved:

(vii) *Under the assumptions of* (vi), *the continuous function* $\phi = \phi(u, v)$ *in* (26), *belonging to an asymptotic Tchebychef parametrization* $X = X(u, v)$ *of* $S$, *is of class* $C'$ *if and only if the corresponding parametrization* $X = X(u, v)$ *is of class* $C''$.

It is clear from (27) that if $X = X(u, v)$ is of class $C''$, then $\phi(u, v)$ is of class $C'$. Thus, only the converse needs a proof. In view of (vi bis), this converse is contained in the following lemma:

LEMMA. *Let* $S$ *be a surface of class* $C''$, *and* $X = X(u, v)$ *a* $C'$*-parametrization of* $S$ *with the properties that the coefficients* $g_{ik} = g_{ik}(u, v)$ *in*

$$|\, dX(u, v)\,|^2 = g_{11}du^2 + 2g_{12}du\,dv + g_{22}dv^2 \tag{45}$$

*are of class* $C'$ *and that* (28) *holds. Then the parametrization* $X = X(u, v)$ *of* $S$ *is of class* $C''$.

This lemma has applications other than (vii), as is indicated by (vi bis) and a similar result ([3], p. 724) concerning geodesic polar coordinates. The proof of the lemma will make it clear that the conclusion $C^n \to C^{n+1}$ can be extended from $n = 1$ to any $n$.

**13. Proof of the Lemma.** It will be shown that $X_{uu}(u, v)$ exists and is continuous. To this end, let

$$\Delta g = \{g(u + h, v) - g(u, v)\}/h,$$

where $g$ is a (scalar or vector) function, $|\, h\,| \neq 0$ is a small number and $u, v$ are fixed, and put

$$A = \tfrac{1}{2}\{X_u(u+h, v) + X_u(u, v)\}, \quad B = X_v(u, v), \quad C = N(u, v).$$

If $|h|$ is sufficiently small, the vectors $A$, $B$, $C$ are linearly independent. Hence, there exist three (unique) scalars $\alpha = \alpha_h(u, v)$, $\beta = \beta_h(u, v)$, $\gamma = \gamma_h(u, v)$ satisfying

$$\Delta X_u = \alpha_h A + \beta_h B + \gamma_h C. \tag{46}$$

If (46) is multiplied (scalarly) by $A$, $B$, $C$, respectively, there results the following system of equations for $\alpha_h$, $\beta_h$, $\gamma_h$:

$$\begin{aligned} A \cdot \Delta X_u &= \alpha_h\{g_{11}(u, v) + o(1)\} + \beta_h\{g_{12}(u, v) + o(1)\} + o(1)\gamma_h, \\ B \cdot \Delta X_u &= \alpha_h\{g_{11}(u, v) + o(1)\} + \quad \beta_h g_{12}, \\ C \cdot \Delta X_u &= \alpha_h o(1) \qquad\qquad\qquad\qquad + \gamma_h, \end{aligned}$$

where the $o(1)$-terms refer to $h \to 0$. The determinant of this system is $\det(g_{ik}) + o(1)$, which does not vanish for sufficiently small $|h|$. Hence, the last three equations can be solved for $\alpha_h$, $\beta_h$, $\gamma_h$. It follows that, as $h \to 0$, the numbers $\alpha_h$, $\beta_h$, $\gamma_h$ tend to limits, $\alpha_0$, $\beta_0$, $\gamma_0$, which are continuous functions of $(u, v)$, if the same is true of $A \cdot \Delta X_u$, $B \cdot \Delta X_u$, $C \cdot \Delta X_u$.

It will be shown that the latter scalar products tend to continuous limits, as $h \to 0$. It is seen from the definition of $g_{11}$ that

$$\tfrac{1}{2}\Delta g_{11} = \tfrac{1}{2}\{X_u(u+h, v) + X_u(u, v)\} \cdot \Delta X_u.$$

It follows that, as $h \to 0$, the scalar product $A \cdot \Delta X_u$ tends to the continuous function $\frac{1}{2}\partial g_{11}/\partial u$, since it is assumed that $g_{11}$ is of class $C'$. Similarly, the identity

$$\Delta g_{12} = X_v(u, v) \cdot \Delta X_u + X_u(u+h, v) \cdot \Delta X_v$$

implies that $B \cdot \Delta X_u$ tends to the continuous function $\partial g_{12}/\partial u - X_u \cdot X_{uv}$, since it is assumed that $g_{12}$ is of class $C'$ and that (28) holds. Finally, the identity $X_u \cdot N \equiv 0$ leads to

$$0 = \Delta(X_u \cdot N) = N(u, v) \cdot \Delta X_u + X_u(u+h, v) \cdot \Delta N.$$

Since $S$ is of class $C''$ in some parametrization, its unit normal vector, which is invariant up to a factor $\pm 1$, is of class $C'$ in any $C'$-parametrization of $S$. Thus, the last identity implies that the scalar product $C \cdot \Delta X_u$ tends, as $h \to 0$, to the continuous function $-X_u \cdot N_u$. It follows that the limits, $\alpha_0$, $\beta_0$, $\gamma_0$, of $\alpha_h$, $\beta_h$, $\gamma_h$ exist and are continuous. Hence, (46) shows that $X_{uu}(u, v)$ exists and has the value

$$\alpha_0 X_u(u, v) + \beta_0 X_v(u, v) + \gamma_0 N(u, v),$$

and is therefore continuous. It is shown in a similar manner that $X_{vv}(u, v)$ exists and is continuous. This completes the proof of the Lemma.

**14. On the smoothness of asymptotic Tchebychef parametrizations.** The remark following (v) and that preceding (vii) will now be proved. These remarks are contained in the following assertion:

(viii) *Let a surface $S$ be of constant negative curvature ($K \equiv -1$) and suppose that $S$ possesses an asymptotic Tchebychef parametrization $X = X(u, v)$ of class $C''$. Then $S$ is of class $C'''$ in a Cartesian parametrization, say $z = z(x, y)$.*

*Proof of* (viii). According to (26), the elements of the first fundamental matrix $(g_{ik}) = (g_{ik}(u, v))$ are given by $g_{11} = g_{22} \equiv 1$ and $g_{12} = \cos\phi$. In particular,

$$\det(g_{ik}) = \sin^2\phi, \text{ hence } \sin\phi \neq 0. \tag{47}$$

The elements of the second fundamental matrix are given by $h_{11} = h_{22} \equiv 0$ and $h_{12} = \pm \sin\phi$. In fact, $h_{11} = h_{22} = 0$ follows from the fact that $u = \text{const.}$ and $v = \text{Const.}$ on asymptotic curves; while $h_{12}^2 = \sin^2\phi$ is equivalent to the identity $-1 \equiv K = \det(h_{ik})/\det(g_{ik})$. (To avoid ambiguity, it can be supposed that $h_{12} = \sin\phi$; otherwise $(u, v)$ can be replaced by the parameters $(-u, v)$, the latter replacement being equivalent to a re-orientation of the asymptotic lines $v = \text{Const.}$)

The derivation formulae of Weingarten for the partial derivative of the unit normal vector $N(u, v)$ supply the relations

$$N_u = (X_u \cos\phi - X_v)/\sin\phi, \qquad N_v = -(X_u - X_v \cos\phi)/\sin\phi,$$

which show that $N(u, v)$ is of class $C''$.

Let the coordinate system $(x, y, z)$ be so chosen that $S$ has a $C''$-parametrization of the form $z = z(x, y)$. Since the vector $N(u, v)$ is $\pm(p, q, -1)\{1 + p^2 + q^2\}^{-\frac{1}{2}}$, where $p = z_x(x, y)$, $q = z_y(x, y)$ are considered as functions of $u, v$, it follows that $p, q$ are of class $C''$ as functions of $u, v$. But the parameter transformation $(x, y) \to (u, v)$ is of class $C''$ (with non-vanishing Jacobian). Hence, $p, q$ are of class $C''$ as functions of $x, y$ also. Consequently, $z = z(x, y)$ is a function of class $C'''$. This proves (viii).

**15. On the Theorema Egregium.** If Tchebychef parameters are used on a surface having the Gaussian curvature $K \equiv -1$, then the Theorema Egregium is known to reduce to

$$\phi_{uv} = \sin\phi, \tag{48}$$

or to an integrated form which is the standard formula of Hazzidakis,

$$[\phi] = \int_{v_1}^{v_2}\int_{u_1}^{u_2} \sin\phi(u,v)\,du\,dv, \tag{49}$$

where

$$[\phi] = \phi(u_1, v_1) - \phi(u_2, v_1) + \phi(u_2, v_2) - \phi(u_1, v_2), \tag{50}$$

$\phi = \phi(u, v)$ being the function occurring in (26) (cf., e. g., the presentation [1], § 60-§ 63 and § 245). But as mentioned before (vi), the classical theory of pseudo-spheres $S$ tacitly assumes that $S$ has an artificially high degree of smoothness. Correspondingly, the standard proof of (48) or (49) depends on the assumption $C''''$. It turns out, however, the drastic reduction, $C'''' \to C''$, mentioned before (vi), is possible for the integrated form (49) also, while the reduction $C'''' \to C'''$ is possible for (48):

(ix) *Under the assumptions and in the notations of* (vi), *the function* $\phi = \phi(u, v)$ *occurring in* (26) *satisfies the identity* (49).

*Proof of* (ix). According to van Kampen's proof [5] of the theorem of Gauss-Bonnet, the latter theorem holds for surfaces of class $C''$ and for contours consisting of a finite number of arcs of class $C''$. In addition, it is clear from van Kampen's proof that, on surfaces of class $C''$, the Jordan contour can be allowed to consist of a finite number of arcs of class $C'$, if the integral of the geodesic curvature over the contour (i. e., the first term is van Kampen's formula (28), p. 134) is replaced by the sum $\Sigma\Delta\theta$. In this sum, extended over all arcs of class $C'$ which constitute the contour, $\Delta\theta$ denotes the oriented variation of a continuous determination of $\theta$ along one of the arcs of class $C'$, and $-\theta$ is the oriented angle from the tangent vector of the arc to a vector transported by Levi-Civita's parallelism along the arc (cf. ibid., p. 131).

Let $R$ be the positively oriented boundary of the small rectangle considered before (34). Then if the preceding form of the theorem of Gauss-Bonnet is applied to the Jordan contour $R$, and if $Q$ denotes the interior of $R$, it follows that

$$\Sigma\,\Delta\theta + \int_Q\int KdS = 2\pi - \Sigma\,\alpha, \tag{51}$$

where $\Sigma\,\alpha$ is the sum of the oriented exterior angles, $\alpha$, at the vertices of the image of $R$ on the surface $S$.

It is understood that $dS = |\,dS\,|$ in (51) denotes the surface element

on $S$. Hence $dS = (\det g_{ik})^{\frac{1}{2}} du\, dv$, where the square root is positive, and $g_{11} = 1$, $g_{12} = \cos\phi$, $g_{22} = 1$, by (26). Since $K \equiv -1$, it follows that

$$\iint_Q K dS = -\int_{v_1}^{v_2}\int_{u_1}^{u_2} |\sin\phi(u, v)|\, du\, dv,$$

$Q$ being the interior of the rectangle $R$ having the vertices $(u_i, v_k)$, where $u_1 < u_2$, $v_1 < v_2$. Thus (51) reduces to

$$\Sigma\, \Delta\theta - \int_{v_1}^{v_2}\int_{u_1}^{u_2} \sin\phi(u, v)\, du\, dv = 2\pi - \Sigma\, \alpha, \tag{52}$$

if it is assumed that $|\sin\phi| = \sin\phi$. This assumption is admissible, since $\sin\phi(u, v)$ is a continuous function which cannot vanish, by (47), and which goes over into $-\sin\phi$ if $(u, v)$ is replaced by $(-u, v)$ (the latter replacement is equivalent to a re-orientation of the asymptotic lines $v = \text{Const.}$).

Next, the orientation of the rectangle $R$ and the definition of the angles $\alpha$, given after (51), show that

$$\Sigma\, \alpha = (\pi - \phi^1) + \phi^2 + (\pi - \phi^3) + \phi^4, \tag{53}$$

where

$$\phi^1 = \phi(u_1, v_1), \quad \phi^2 = \phi(u_2, v_1), \quad \phi^3 = \phi(u_2, v_2), \quad \phi^4 = \phi(u_1, v_2). \tag{54}$$

In order to evaluate the remaining sum, $\Sigma\Delta\theta$, which occurs in (52), use will be made of the fact that, according to (27), both vectors $X_u = X_u(u, v)$, $X_v = X_v(u, v)$ are of unit length. On the other hand, by (29) or the end of the proof of (vi), the vector $X_{uv}$ is parallel to the normal vector, $N$, of $S$. Finally, two sides of the rectangle $R$ are formed by asymptotic curves $u = \text{const.}$, and the other two by asymptotic curves $v = \text{Const.}$ Consequently, the vectors $X_v$, $X_u$ represent parallel transportations of themselves along the former and latter two sides of $R$, respectively. It follows therefore from the definition of the terms of the sum $\Sigma\Delta\theta$, given before (51), that, along the sides of $R$, the angle $\theta$ can be chosen to be $-\phi$ or $\phi$ according as $v = \text{Const.}$ or $u = \text{const.}$ Consequently, from (53),

$$\Sigma\Delta\theta = (\phi_1 - \phi_2) + (\phi_3 - \phi_2) + (\phi_3 - \phi_4) + (\phi_1 - \phi_4). \tag{55}$$

Clearly, (55) can be contracted into $\Sigma\Delta\theta = 2[\phi]$, and (53) into $\Sigma\alpha = 2\pi - [\phi]$, where $[\phi]$ is defined by (50). Hence, (52) reduces to (49). This completes the proof of (ix).

## PART IV. Lines of Curvature.

**16. On lines of curvature ($K \equiv -1$).** The following analogue of (ii) will now be proved:

(x) *If a surface, $S$, of class $C''$ is of constant negative curvature, then either of the directions of principal curvature at a point, $P$, of $S$ determines a unique line of curvature through $P$.*

Clearly, (x) implies that $S$ has a (locally) schlicht parametrization, $X = X(u, v)$, in which $X(u, v)$ is a continuous vector function having the property that $u = \text{const.}$ and $v = \text{Const.}$ represent lines of curvature, and conversely. Actually, it turns out that such a parametrization, $X = X(u, v)$, of $S$ can be chosen not only to be continuous but of class $C'$ as well. In other words, (iv) and (vi) admit of the following analogue:

(xi) *Let a surface $S$ be of constant negative curvature ($K \equiv -1$) and of class $C''$. Then there exists, near every point of $S$, a $C'$-parametrization $X = Y(u, v)$ in which $u = \text{const.}$ and $v = \text{Const.}$ are lines of curvature.*

*Furthermore, such a parametrization can be so chosen that the squared line-element on $S$ becomes of the form*

$$|dY(u, v)|^2 = g\,du^2 + (1 - g)^{\frac{1}{2}}dv^2, \qquad 0 < g < 1, \tag{56}$$

*where the function $g = g(u, v)$ is continuous.*

*Proof of* (x) *and* (xi). If $P$ is a point of a surface $S$ of class $C''$ and of negative curvature, Dupin's indicatrix shows that the directions of principal curvature bisect the angles between the asymptotic directions. In particular, if $X = X(u, v)$ denotes, near $(u, v) = (0, 0)$, the local $C'$-parametrization supplied by (vi), and if $Y(u, v)$ is defined by

$$Y(u, v) = X(\tfrac{1}{2}(u + v), \tfrac{1}{2}(u - v)), \tag{57}$$

then, since $X = X(\text{const.}, v)$ and $X = X(u, \text{Const.})$ are asymptotic curves, $X = Y(\text{const.}, v)$ and $X = Y(u, \text{Const.})$ are lines of curvature (and conversely). Hence, those assertions of (xi) which do not involve $g$ and (56) are equivalent to the particular case $K \equiv -1$ of (iv). Correspondingly, (xi) follows from (ii) by virtue of the $C'$-parametrization used in (57).

**17. A uniqueness theorem for lines of curvature.** There is little doubt that the possibility of non-uniqueness, established by (i) for the asymptotic curves on surfaces of class $C''$, remains realizable if the asymptotic curves are

replaced by the lines of curvature and, correspondingly, the assumption $K < 0$ by the assumption that $S$ be free of umbilical points; that is, of points at which $h_{11} : h_{12} : h_{22} = g_{11} : g_{12} : g_{22}$. As is well-known, the exclusion of this proportionality is equivalent to the exclusion of the sign of equality in the inequality $H^2 \geqq K$, which is satisfied by the mean curvature, $H$, and the Gaussian curvature, $K$, at every point of any surface of class $C''$.

Clearly, what would formally correspond to (iii) is the following assertion: If $S$ is a surface of class $C^{**}$ containing no umbilical point, then every point of $S$ issues only two lines of curvature (one in each of the directions of principal curvature). Actually, this formal transfer of (iii) to the case of lines of curvature will remain problematic. In fact, what corresponds to (iii) turns out to be a criterion requiring that *both* $H$ and $K$ be subject to Lipschitz's condition (in a $C''$-parametrization of $S$), whereas (iii) assumes a Lipschitz condition only for $K$; cf. the definition of the class $C^{**}$, given before (iii). In other words, the adaptation of the device used in the proof of (iii) leads only to the following criterion:

(xii) *If $S : z = z(x, y)$ is of class $C''$ and if both curvatures, $H = H(x, y)$ and $K = K(x, y)$, of $S$ satisfy uniform Lipschitz conditions in $x$ and $y$ together, then every non-umbilical point of $S$ issues a unique pair of lines of curvature.*

**18. Proof of (xii).** First, the exclusion of umbilical points means that $H^2 - K > 0$. On the other hand, the differential equation defining lines of curvature on a surface $z = z(x, y)$ of class $C''$ can be written in the form

$$d\{p/(1 + p^2 + q^2)^{\frac{1}{2}}\} = \{H \pm (H^2 - K)^{\frac{1}{2}}\}dx, \tag{58}$$

where $p = z_x$, $q = z_y$ and

$$H = \{2pqs - (1 + p^2)t - (1 + q^2)r\}/(1 + p^2 + q^2)^{3/2},$$

$$K = (rt - s^2)/(1 + p^2 + q^2)^2$$

are thought of as expressed as functions of $(x, y)$. In fact, (58) is just a restatement of the equation of Rodrigues ($2H$ being the sum, and $K$ the product, of the principal curvatures).

Put

$$m = p/(1 + p^2 + q^2)^{\frac{1}{2}}. \tag{59}$$

Then it is clear that both equations (58) are of the form

$$dm = f(x, y)\,dx, \tag{60}$$

where $f(x, y)$ satisfies a Lipschitz condition (in $x$ and $y$ together), since the same is true of both $H(x, y)$ and $K(x, y)$. Hence it is clear that the argument used at the end of the proof of (iii) will become applicable to (60) if it is ascertained that the Jacobian of the binary $C'$-transformation $(x, y) \to (x, m)$ is distinct from 0.

According to (59), this Jacobian is identical with the partial derivative $\partial\{p/(1 + p^2 + q^2)^{\frac{1}{2}}\}/\partial y$. The non-vanishing of the latter at a point $(x, y)$ is readily verified if it is assumed that the directions of the coordinate axes $x, y$ do not correspond to the directions of principal curvature at that point (and therefore near that point). Finally, this assumption can always be satisfied by a suitable rotation of the $(x, y)$-plane. Clearly, this step corresponds to the normalization (5) in the proof of (iii).

## PART V. Torses.

**19. Asymptotic lines ($K \equiv 0$).** Suppose that a surface $S$ of class $C''$ is a torse, i. e., that $K \equiv 0$. If a point $P$ of $S$ is not a flat point, that is, a point at which the second fundamental matrix $(h_{ik})$ vanishes, then, since $K \equiv 0$ means that $\det(h_{ik}) \equiv 0$, the point $P$ determines a unique asymptotic direction $du_1 : du_2$ satisfying (1). There arises the question whether $P$ can issue more than one asymptotic curve. It turns out that, as in the case $K \equiv -1$ treated in (ii), this possibility cannot arise:

(xiii) *If a surface, $S$, of class $C''$ is a torse, then every non-flat point, $P$, of $S$ issues a unique asymptotic curve. The asymptotic curves are straight lines, along which the tangent plane is invariable, and are the generators of the torse $S$.*

This assertion is a consequence of the considerations in [4], p. 770. Those considerations also show that if $S$ is given in the form $z = z(x, y)$ for small $x^2 + y^2$, and if $P$: $(x, y) = (0, 0)$ is not a flat point and therefore, without loss of generality, $z_{yy}(0, 0) \neq 0$, then the asymptotic curves and only these are determined by the equation $q = \text{const.}$, where $q = z_y(x, y)$; cf. (9) above.

**20. Asymptotic parametrizations ($K \equiv 0$).** Theorem (xiii) and its proof lead to the natural parametrization of torses:

(xiv) *If $S$ is a torse of class $C''$ containing no flat point, then every sufficiently small piece of $S$ possesses a $C'$-parametrization of the form*

$$X(u, v) = A(u)v + B(u), \qquad (A(u) \neq 0), \tag{61}$$

*where $X$ denotes the vector $(x, y, z)$, and $A(u)$, $B(u)$ are vector functions of class $C'$ satisfying*

$$\det(A, A', B') \equiv 0 \qquad (' = d/du). \tag{62}$$

*Proof of* (xiv). If $S$ is given in the form $z = z(x, y)$, introduce, instead of $x, y$, the parameters $x, q$ defined by the $C'$-transformation (8). The latter has a non-vanishing Jacobian near the origin, since (5) can be assumed. Thus the resulting parametrization, say $X = X(q, x)$, of $S$ is of class $C'$. Furthermore, by (xiii) and the remark following it, all asymptotic curves and only these are given by $q = \text{const}$. Accordingly, the direction of the vector $\partial X(q, x)/\partial x$ is independent of $x$. But the first component of $X(q, x)$ is $x$; hence, the first component of $\partial X(q, x)/\partial x$ is 1. Consequently, $\partial X(q, x)/\partial x$ is independent of $x$. It follows therefore by a quadrature that $X(q, x)$ appears in the form (61) by placing $u = q$, $v = x$. Furthermore, $A \neq 0$, since $A = \partial X/\partial v$. In addition, $A(u)$ and $B(u)$ are of class $C'$, since $X(u, v)$ is. Finally, (62) is clear from the fact that the normal vector is constant along the asymptotic lines $u = \text{const}$. (that is, from the fact that the vector product of $A'v + B'$ into $A$ is parallel to the vector product of $B'$ into $A$). This proves (xiv).

**21. On the smoothness of the parametrization (61).** Although (xiv) is an analogue of (vi), the assertion of (viii) does not have an analogue. In fact, if $f(x)$ is a function of class $C''$ for $-1 < x < 1$ and $S$ is the surface given by $z = z(x, y) = f(x)y$,. then $S$ is a torse of class $C''$. It possesses a parametrization (61), where $x = u$, $v = y$ and $A(u) = (1, 0, f(u))$, $B(u) = (0, 0, 0)$. This parametrization of $S$ is of class $C''$, but is not of class $C'''$ unless $f(x)$ is.

Thus there arises the question as to whether or not (v) has an analogue. The answer is affirmative:

(xv) *Under the assumptions of* (xiv), *let $S: X = (x, y, z)$ be parametrized (locally) in the form* (61). *Then no such parametrization need be of class $C''$.*

**22. Proof of (xv).** Let $f(u)$ be a function of class $C''$ on the interval $-1 \leqq u \leqq 1$, and put $X = X(u, v)$, where, if $X = (x, y, z)$,

$$x = v, \qquad y = -f'(u)v + u \tag{63}$$

and

$$z = \{f(u) - uf'(u)\}v + \tfrac{1}{2}u^2. \tag{64}$$

Clearly, the Jacobian $\partial(x, y)/\partial(u, v)$ is $-1$ at $(u, v) = (0, 0)$. Hence, in a vicinity of $(u, v) = (0, 0)$, the surface (63)-(64) has an equation of the form $z = z(x, y)$. It will be shown that this surface, $S$, is of class $C''$ in its $(x, y)$-parametrization; that $S$ is a torse; and that the point $P$ corresponding to $(x, y) = (0, 0)$ is not a flat point (as is not, therefore, any point sufficiently close to $P$).

A simple calculation involving the Jacobian matrix of the transformation (63) shows that

$$z_x = f(u) \text{ and } z_y = u, \tag{65}$$

where $u = w(x, y)$. Since $u$ is a function of class $C'$ in $(x, y)$, and $f$ is a function of class $C''$ in $u$, it follows that $z_x, z_y$ are functions of class $C'$ in $(x, y)$. Hence, $S$ is of class $C''$ in terms of its $(x, y)$-parametrization.

It follows from (65) that $z = z(x, y)$ satisfies the partial differential equation $rt - s^2 = 0$; so that, by (3), the Gaussian curvature of $S$ vanishes identically. It is also seen from (65) and (63) that $z$ satisfies the normalization (5); so that $(x, y) = (0, 0)$ is not a flat point.

In order to complete the proof of (xv), it remains to show that if $f(u)$ is a suitably chosen function of class $C''$, then no $C''$-parametrization of any vicinity of $P$: $(0, 0)$ on $S$ is of the form (61). It turns out that $S$ must have this property whenever $f(u)$ fails to have a third derivative at $u = 0$.

First, (63)-(64) is a $C'$-parametrization of the form (61). Let $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$ be a change of parameters which transforms (61), that is, (63)-(64), into a parametrization of the same form, say

$$X(\alpha; \beta) = L(\alpha)\beta + M(\alpha). \tag{61 bis}$$

Then, since $w = \text{const.}$ and $\alpha = \text{Const.}$ are asymptotic curves, $u(\alpha, \beta)$ is a function of $\alpha$ alone, say $u = u(\alpha)$. For a fixed $u$ or $\alpha$, the variables $v$ or $\beta$ are, up to a translation and a change of scale, the arc-lengths along the asymptotic curves. Hence $v(\alpha, \beta)$ is of the form $g(\alpha)\beta + h(\alpha)$, where $g(\alpha) \neq 0$. Consequently, a necessary condition for a transformation $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$ taking (63)-(64) into the form (61 bis) is the following structure:

$$u = u(\alpha), \qquad v = g(\alpha)\beta + h(\alpha); \qquad g(\alpha) \neq 0. \tag{66}$$

Suppose, if possible, that (63)-(64) can be transformed into (61 bis) in such a way that $(\alpha, \beta) = (0, 0)$ corresponds to $(u, v) = (0, 0)$ and that the resulting vector function $X(\alpha; \beta)$ in (61 bis) is of class $C''$ in a vicinity of $(\alpha, \beta) = (0, 0)$. If $X = (x, y, z)$, the first of the equations (63) shows

that $x = x(\alpha;\beta)$ is identical with $g(\alpha)\beta + h(\alpha)$. It follows that $g(\alpha)$ and $h(\alpha)$ are of class $C''$ near $\alpha = 0$. On the other hand, the second of the equation (63) shows that $y = y(\alpha;\beta)$ reduces to

$$-f'(u(\alpha))g(\alpha)\beta - f'(u(\alpha))h(\alpha) + u(\alpha).$$

Hence, $f'(u(\alpha))g(\alpha)$ and $-f'(u(\alpha))h(\alpha) + u(\alpha)$ are of class $C''$ near $\alpha = 0$. Since $g(\alpha) \not\equiv 0$, it follows that $f'(u(\alpha))$, and therefore $u(\alpha)$, is of class $C''$ near $\alpha = 0$.

Thus the three functions of $\alpha$ which occur in (66), namely, $u(\alpha)$, $g(\alpha)$ and $h(\alpha)$, are of class $C''$. In order that (61 bis) be an admissible parametrization of $S$, it is necessary that the Jacobian $\partial(u, v)/\partial(\alpha, \beta)$ be distinct from 0 at $(\alpha, \beta) = (0, 0)$. This Jacobian is $g(\alpha)du(\alpha)/d\alpha$, by (66). But if $f(u)$ fails to have a third derivative at $u = 0$, then, since $du(\alpha)/d\alpha \not\equiv 0$ at $\alpha = 0$, the function $f'(u(\alpha))$ of $\alpha$ cannot have a second derivative at $\alpha = 0$. This completes the proof of (xv).

**23. On lines of curvature ($K \equiv 0$).** The following analogue of (x) will now be proved:

(xvi) *If $S$ is a torse of class $C''$, then either of the directions of principal curvature at a non-flat point, $P$, of $S$ determines a unique line of curvature.*

*Proof of* (xvi). The simple parametrization of $S$ supplied by (xiv) cannot be inserted into the differential equations of the lines of curvature, since those differential equations assume a parametrization of class $C''$, whereas (61) is of class $C'$ only. Let, therefore, $S$ be given in a $C''$-parametriaztion, say as $z = z(x, y)$. Then the lines of curvature are defined by

$$dN + \lambda dX = 0 \tag{67}$$

(Rodrigues), where $N$ denotes the unit normal vector, $X$ the vector $(x, y, z)$, finally $\lambda$ either of the two principal curvatures.

The assumption $K \equiv 0$ implies that one of the principal curvatures vanishes identically. On the other hand, the assumption that $P$, and hence every point in a sufficiently small vicinity of $P$, is a non-flat point ensures that the other principal curvature is distinct from zero. For the first of these two determinations of $\lambda$, it is seen from (67) that $dN = 0$, i. e., $N = \text{const}$. Hence, in the direction of the principal curvature $\lambda = 0$ every point issues exactly one line of curvature, and this line is an asymptotic curve also; cf., e. g., [4], p. 770.

Since the two directions of principal curvature at a given point are orthogonal, the remaining case, $\lambda \not\equiv 0$, of (67) is equivalent to the differential equations of the orthogonal trajectories of the sheaf of the asymptotic curves. But in terms of the parametrization (61) of $S$, the asymptotic curves are represented by $u = \text{const}$. Hence, the differential equation of the orthogonal trajectories in question is

$$g_{12}(u, v) + g_{22}(u, v)\, dv/du = 0. \tag{68}$$

It is readily verified from (61) that (68) reduces to

$$(A \cdot A')v + (A \cdot B') + (A \cdot A)\, dv/du = 0, \qquad (' = d/du), \tag{69}$$

where the dot denotes scalar multiplication. Since (69) is a linear differential equation for $v = v(u)$, with continuous functions of $u$ as coefficients, the solutions of (69) are uniquely determined by their initial conditions. This proves (xvi).

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# ON THE SMALL DIVISORS IN INTEGRATIONS BY LAPLACE TRANSFORMS.*

By AUREL WINTNER.

**1.** If $\alpha$ is a matrix, of $m$ rows and $m$ columns, let $|\alpha|$ denote the greatest of the absolute values of its $m^2$ elements. Correspondingly, if $\alpha = \alpha(s)$, where $0 \leqq s < \infty$, let

$$[\alpha] = \int_0^\infty |d\alpha(s)| \tag{1}$$

be the greatest of the total variations of the $m^2$ scalar functions which constitute the matrix function $\alpha(t)$. Thus the matrix

$$A(t) = \int_0^\infty e^{-ts} d\alpha(s) \tag{2}$$

consists of $m^2$ Laplace integrals which are absolutely-uniformly convergent for $0 \leqq t < \infty$ if $[\alpha] < \infty$. If in addition

$$\int_0^\infty |d\alpha(s)|/s < \infty, \tag{3}$$

then it is clear from (2) that

$$\int_0^\infty |A(t)|\, dt < \infty. \tag{4}$$

Needless to say,

$$\alpha(+0) = \alpha(0) \tag{5}$$

is necessary for (3). The following theorem will be proved:

In the linear system of differential equations

$$x' = A(t)x, \qquad (' = d/dt), \tag{6}$$

where $x$ is a vector $(x_1, \cdots, x_m)$, let the coefficient matrix $A(t)$ be representable, for $0 \leqq t < \infty$, in the form (2), where $[\alpha] < \infty$. Suppose further

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that $\alpha$ satisfies (3). Then (6) has $m$ linearly independent solution vectors $x = x^1(t), \cdots, x = x^m(t)$ which, when ordered in the form of a matrix $X = (x^1, \cdots, x^m)$, are representable, for $0 \leqq t < \infty$, in the form

$$(7) \quad X(t) = \int_0^\infty e^{-st}\, d\beta(s), \qquad \text{where } [\beta] < \infty \text{ and } \beta(+0) - \beta(0) = E$$

($E$ denotes the unit matrix); so that, by (7),

$$(8) \quad X(t) = E + X_0(t), \text{ where } X_0(t) \to 0$$

as $t \to \infty$, since

$$(9) \quad X_0(t) = \int_{+0}^\infty e^{-ts}\, d\beta(s).$$

**2.** Condition (3) is automatically satisfied if the range of integration in (2) does not reach down to $s = 0$, that is, if $\alpha(s) = \text{const.}$ for small non-negative $s$. For this particular case, the theorem was proved in [2]. The method applied there fails if the points of non-constancy of $\alpha(s)$ are allowed to reach down to $s = 0$, since then the successive approximations lead to "small divisors," substantially of the same type as those occuring in celestial mechanics. Correspondingly, the content of the above theorem is to the effect that the "small divisors" are harmless as long as the "frequency function," $\alpha(s)$, of (2) is subject to the restriction (3).

It turns out that (3) is the best possible condition of its type. In order to see this, suppose that $m = 1$, i. e., that $A(t)$, $\alpha(s)$, $x = X$ are scalars, and let $\alpha(s)$ be non-decreasing and bounded for $0 \leqq s < \infty$. Then $[\alpha] < \infty$ is satisfied, and (2) shows that the general solution of (6) is a constant multiple of

$$x(t) = \exp \int_0^\infty (1 - e^{-st})/s\, d\alpha(s).$$

Clearly, this $x(t)$ tends, as $t \to \infty$, to $\infty$ whenever the case $|d\alpha(s)| = d\alpha(s)$ of the assumption (3) is violated. Hence, nothing like (7) can hold, in general, unless (3) is assumed.

**3.** The proof of the theorem announced in §1 proceeds as follows:

Each of the $m$ columns of a matrix $X(t)$ is a solution vector $x(t)$ of (6) if

$$(10) \quad X'(t) = A(t)X(t),$$

and these $m$ solutions will be linearly independent if $\det X(t) \neq 0$ holds for some $t$ (in which case it is known to hold for every $t$). But if the corollary (8) of (7) is satisfied, then $\det X(t) \neq 0$ for large $t$ (hence, for every $t$). On the other hand, (7) reduces (10) to

$$X(t) = E - \int_t^\infty A(u)X(u)\,du. \tag{11}$$

Hence, what is sought for is a solution $X(t)$ of (11) which is of the form (7).

To this end, try successive approximations, by placing

$$X(t) = E + \sum_{n=1}^{\infty} X_n(t), \tag{12}$$

where, in accordance with (11), the terms $X_n(t)$ are assigned by the recursion formula

$$X_{n+1}(t) = -\int_t^\infty A(u)X_n(u)\,du, \tag{13}$$

with

$$X_1(t) = -\int_t^\infty A(u)\,du. \tag{14}$$

In order that (12) should become of the form (7), it is natural to assume that

$$X_n(t) = \int_0^\infty e^{-ts}\,d\beta_n(s), \tag{15}$$

where $[\beta_n] < \infty$ and

$$\beta_n(+0) = \beta_n(0) \qquad \text{(and, e. g., } \beta_n(0) = 0\text{)}. \tag{16}$$

On the other hand, direct substitutions show (cf. [2]) that, by virtue of (15) and (2), the relation (14) is formally identical with

$$-s\,d\beta_1(s) = d\alpha(s), \qquad (0 < s < \infty), \tag{17}$$

and the recursion formula (13) with

$$-s\,d\beta_{n+1}(s) = d\alpha(s) * \beta_n(s), \qquad (0 < s < \infty), \tag{18}$$

where the asterisk denotes the operation of convolution (cf. [2]).

It will be proved that if both integrals (1), (3) are finite, then (17), (18) and (16) define, on the closed half-line $0 \leqq s < \infty$, a sequence of

matrix functions $\beta_n(s)$ each of which is of finite total variation, $[\beta_n] < \infty$, and that the sum of these total variations is finite,

$$\sum_{n=1}^{\infty} [\beta_n] < \infty. \tag{19}$$

If this is proved, it will follow that (7) and (11) become satisfied by placing

$$\beta(s) = \sum_{n=1}^{\infty} \beta_n(s) \text{ if } 0 < s < \infty, \tag{20}$$

and $\beta(0) = \beta(+0) - E$. In fact, the deduction of (7) and (11) will then be the same as in [2] and will, therefore, be omitted.

**4.** If $\alpha_{ik}(s)$, where $i = 1, \cdots, m$ and $k = 1, \cdots, m$, denote the elements of the matrix $\alpha(s)$, put

$$\lambda(s) = m \max \int_0^s |d\alpha_{ik}(u)|, \qquad (0 \leqq s < \infty), \tag{21}$$

where the symbol max refers to $i$ and $k$ together. Thus $\lambda(s)$ is a non-decreasing scalar function. Hence it is clear that the scalar recursion formula

$$s d\mu_{n+1}(s) = d\{\lambda(s) * \mu_n(s)\}, \text{ where } 0 < s < \infty, \tag{22}$$

will be a majorant of the matrix recursion formula (18) if, corresponding to (17),

$$s d\mu_1(s) = d\lambda(s), \text{ where } 0 < s < \infty. \tag{23}$$

In fact, since $d\lambda(s) \geqq 0$, it is clear from (23) and (22) that $d\mu_n(s) \geqq 0$, if $\mu_n(s)$ can be formed at all. Hence, in order to prove the existence of the matrix function $\beta_n(s)$ and the convergence of the series (19), it is sufficient to show that (23) and (22) lead to (non-decreasing) scalar functions $\mu_n(s)$ satisfying

$$\sum_{n=1}^{\infty} [\mu_n] < \infty, \text{ where } [\mu_n] = \int_0^{\infty} d\mu_n(s) \qquad (d\mu_n(s) \geqq 0). \tag{24}$$

The assumptions to be used are

$$d\lambda(s) \geqq 0; \qquad \lambda(0) = 0, \qquad \lambda(\infty) < \infty \tag{25}$$

and

$$\int_0^{\infty} s^{-1} d\lambda(s) < \infty \tag{26}$$

(hence $\lambda(+0) = \lambda(0)$). In fact, it is clear from (21) that (25), (26) are equivalent to the convergence of the respective integrals (1), (3).

The proof will be based, not directly on (22) and (23), but on the circumstances that

$$y_{n+1}(t) = \int_t^\infty a(s) y_n(s) ds \tag{27}$$

and

$$y_1(t) = \int_t^\infty a(u) du \tag{28}$$

correspond to (22) and (23) in the same way as (13) and (14) correspond to (18) and (17) if, corresponding to (2) and (15),

$$a(t) = \int_0^\infty e^{-ts} d\lambda(s) \tag{29}$$

and

$$y_n(t) = \int_0^\infty e^{-ts} d\mu_n(s). \tag{30}$$

Then

$$y(t) = 1 + \sum_{n=1}^{\infty} y_n(t) \tag{31}$$

corresponds to (12), and

$$y(t) = 1 + \int_t^\infty a(u) y(u) du \tag{32}$$

to (11).

Accordingly, (27), (28) and (31) represent the process of successive approximations to a solution of (32), where $a(t)$ is a given scalar function, (29). What is at stake is an appropriate convergence of this process of successive approximations.

**5.** First, it will be shown that the functions $\mu_1(s), \mu_2(s) \cdots$ can be defined, for $0 \leqq s < \infty$ and in accordance with (30) and (27)-(28), in such a way that

$$d\mu_n(s) \geqq 0; \qquad \mu_n(0) = 0, \mu_n(\infty) < \infty. \tag{33}$$

In view of (30), this will prove that all functions $y_n(t)$ are defined for $0 \leqq t < \infty$.

Insertion of (29) into (28) gives

$$y_1(t) = \int_t^\infty \int_0^\infty e^{-us} d\lambda(s)\, du = \int_0^\infty e^{-ts} s^{-1} d\lambda(s),$$

the last rearrangement being justified by (25) and (26). Accordingly, (30) holds for $n = 1$ if $d\mu_1(s) = s^{-1} d\lambda(s)$. In view of (25) and (26), this means that there exists a $\mu_n(s)$ satisfying the case $n = 1$ of (33). Furthermore, by the parenthetical remark made after (26),

$$\mu_n(+0) = \mu_n(0) \tag{34}$$

holds for $n = 1$.

Suppose that $\mu_n(s)$ has been defined, for a fixed $n$, in such a way that (33) and (34) are satisfied. Then (30) defines for $0 \leqq t < \infty$ a function $y_n(t)$ which is continuous and bounded (even if (34) is not used). Hence

$$\int_0^\infty |\, a(u) y_n(u)\,|\, du < \infty \tag{35}$$

provided that $\int_0^\infty |\, a(u)\,|\, du < \infty$. But (29), where $d\lambda(s) \geqq 0$, shows that $a(t)$ is non-negative. Hence the proviso, on which (35) is based, follows from the convergence of (28), which was verified above.

Thus (35) is true. Hence, (27) defines, for $0 \leqq t < \infty$, a function $y_{n+1}(t)$. The explicit form $y_{n+1}(t)$ follows by inserting (29) and (30) into (27). This gives

$$y_{n+1}(t) = \int_t^\infty \int_0^\infty e^{-us} d_s \{ \int_0^\infty \lambda(s - v)\, d\mu_n(v) \}\, du. \tag{36}$$

In fact, the rearrangement leading to (36) is justified by Fubini's theorem, since (35) is satisfied and $d\lambda \geqq 0$, $d\mu_n \geqq 0$. Finally, one more application of Fubini's theorem shows that (36) can be rearranged into

$$y_{n+1}(t) = \int_0^\infty e^{-ts} d\mu_{n+1}(s), \tag{37}$$

with

$$d\mu_{n+1}(s) = s^{-1} d_s \{ \int_0^\infty \lambda(s - v)\, du_n(v) \}, \tag{38}$$

where the convergence of the integrals (37) and (38), for $0 \leqq t < \infty$ and $0 < s < \infty$ respectively, is part of the statement.

The representation (37) of $y_{n+1}(t)$ means that (30) holds if $n$ is replaced by $n+1$. Since (37) converges at $t=0$, it also follows that

$$\int_0^\infty d\mu_{n+1}(s) < \infty, \text{ where } d\mu_{n+1}(s) \geqq 0 \tag{39}$$

(the second assertion of (39) follows from (38), where $d\lambda \geqq 0$ and $d\mu_n \geqq 0$). If a suitable constant is added to $\mu_{n+1}(s)$, it is seen from (39) that (33) remains true if $n$ is replaced by $n+1$. Finally, (39) and the presence of the factor $s^{-1}$ in (38) imply that (34) must hold when $n$ is replaced by $n+1$.

This completes the induction from $n$ to $n+1$. Its result can be summarized as follows: Starting with (28), the recursion formula (27) defines a sequence of functions $y_n(t)$ which are representable, for $0 \leqq t < \infty$ and $n = 1, 2, \cdots$, in the form (30), where the functions $\mu_n(s)$ satisfy conditions (22) and (34).

What remains to be proved is that the functions $\mu_n(s)$, thus defined, are such as satisfy (24). This will complete the proof of the theorem announced in §1.

Since $d\mu_n(s) \geqq 0$, it is clear from (30) that the assertion is equivalent to the absolute convergence of the series (31) at the point $t=0$. But $a(t) \geqq 0$, by (29), where $d\lambda(s) \geqq 0$. Hence it is seen from (27) and (28) that

$$|y_n(t)| = y_n(t) \leqq y_n(0), \qquad 0 \leqq t < \infty, \tag{40}$$

and it is also clear that

$$y_{n+1}'(t) = -y_n(t)a(t), \qquad y_{n+1}(\infty) = 0, \tag{41}$$

where $n \geqq 0$ and

$$y_0(t) \equiv 1. \tag{42}$$

But an induction leads from (42) and (41) to

$$y_n(t) = \left(\int_t^\infty a(s)\,ds\right)^n / n!\,. \tag{43}$$

Since application of (43) at $t=0$ shows that $\Sigma y_n(0) < \infty$, it follows from the remark made before (40) that the proof is complete.

**6.** It is clear from the above proof of the theorem announced in §1, that the addition rule of "spectra," used in [2], holds in the present case also, and that the same is true of "point spectra"; cf. [1]. Thus, if the theorem of §1 is applied to the case of an $\alpha(s)$ which is a step-function, say

$$\alpha(s) = \sum_{\gamma_n < s} A_n, \text{ where } \gamma_n > 0, \tag{44}$$

then there results the following corollary:

Let $A(t)$ be a matrix function of the form

$$A(t) = \sum_{n=1}^{\infty} A_n e^{-\gamma_n t}, \qquad 0 \leqq t < \infty, \tag{45}$$

where every $\gamma_n$ is positive, every $A_n$ is a constant matrix, and

$$\sum_{n=1}^{\infty} |A_n| < \infty, \qquad \sum_{n=1}^{\infty} |A_n|/\gamma_n < \infty. \tag{46}$$

Without loss of generality, suppose that, by an insertion of amplitude matrices $A_j, A_k, \cdots$ all of which are the zero matrix, the notation in (44), (45) is chosen as follows: If $l$ and $h_1, h_2, \cdots, h_l$ are arbitrary positive integers, then the sum $h_1\gamma_1 + \cdots + h_l\gamma_l$ occurs as a $\gamma_n$. Then (6) has a complete set of linearly independent solution vectors $x(t)$ which, when written as the consecutive columns of a matrix $X(t)$, are representable in the form

$$X(t) = E + \sum_{n=1}^{\infty} B_n e^{-\gamma_n t}, \qquad 0 \leqq t < \infty, \tag{47}$$

where the $B_n$ are constant matrices satisfying

$$\sum_{n=1}^{\infty} |B_n| < \infty, \tag{48}$$

and $E$ denotes the unit matrix.

In fact, (47) and (48) represent (7), while (46) represents (1) and (3) together, since (2) reduces to (45) by virtue of (44).

It may finally be mentioned that, since the proof is based on majorizations, everything remains true if the real half-line $t \geqq 0$ is replaced by the complex half-plane $\Re t \geqq 0$. In particular, the real half-line $t \geqq 0$ can be replaced by the imaginary axis, $\Re t = 0$. Accordingly, the above theorem remains true if, both in (45) and (47), one reads $it$ and $-\infty < t < \infty$ instead of $t$ and $0 \leqq t < \infty$, respectively.

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# ON AN INEQUALITY OF GOLUSIN.*

By JAMES A. JENKINS.[1]

1. Recently Golusin [1], [2] has proved a number of interesting inequalities for univalent functions of which one, in particular, has proved very useful in obtaining bounds for the coefficients of functions univalent in the unit circle. This inequality states that for a function $f(z)$ regular and univalent in $|z|<1$ and normalized at $z=0$ by $f(0)=0$, $f'(0)=1$,

$$|f(z_1)-f(z_2)|/|z_1-z_2| \geqq |f(z_1)f(z_2)|(1-r^2)/r^2, \tag{1}$$

where $|z_1|=|z_2|=r$, $0<r<1$, with equality holding only for

$$f(z)=z/(1+ae^{i\theta}z+e^{2i\theta}z^2),\ \theta=\tfrac{1}{2}(\arg z_1+\arg z_2), \qquad -2\leqq a\leqq 2.$$

The first object of this note is to show that this result is an immediate consequence of a result proved some years ago by Grötzsch [4]. This latter result can, in its turn, easily be proved by the general methods of extremal length or quadratic differentials but these proofs are not essentially different from Grötzsch's original proof by his method of strips.

As Golusin did, we shall derive first an inequality from which (1) follows at once. Let

$$F(\zeta)=\zeta+a_0+a_1/\zeta+\cdots \tag{2}$$

be meromorphic and univalent for $|\zeta|>1$, then we shall show

$$|F(\zeta_1)-F(\zeta_2)|/|\zeta_1-\zeta_2| \geqq 1-1/r^2, \tag{3}$$

where $|\zeta_1|=|\zeta_2|=r$, $r>1$.

The result of Grötzsch in question is as follows: Let $D$ be a domain of the $\zeta$-plane containing the point at infinity and of finite connectivity, let $\zeta_1$, $\zeta_2$ be finite interior points of $D$, then $|F(\zeta_1)-F(\zeta_2)|$ attains its minimum within the class of functions $F(\zeta)$ meromorphic and univalent in $D$ and normalized at infinity by (2) uniquely for the function mapping $D$ on the plane slit along arcs of the confocal family of hyperbolae having the images of $\zeta_1$ and $\zeta_2$ as foci, this being determined up to an arbitrary choice of $a_0$ in (2).

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Now let $D$ be the domain $|\zeta| > 1$. For the proof of (3) it is clearly enough to take $\zeta_1 = re^{i\phi}$, $\zeta_2 = re^{-i\phi}$; $1 < r$, $0 < \phi < \pi$. The function $F^*(\zeta) = \zeta + 1/\zeta$ gives a mapping onto a configuration of the required extremal type. Since

$$|F(\zeta_1) - F(\zeta_2)| = 2(r - 1/r)\sin\phi$$

holds for this $F = F^*$, it follows that

$$|F(\zeta_1) - F(\zeta_2)| \geqq 2(r - 1/r)\sin\phi$$

holds for any $F$. Dividing by $|\zeta_1 - \zeta_2| = 2r\sin\phi$, we obtain (3).

Since the resulting extremal function does not assume the value 0 the result holds also in the more restricted class of functions for which this is true. Other extremal functions in this class are $F^*(\zeta) + a$, $-2 \leqq a \leqq 2$. From this we obtain at once inequality (1) and the indicated extremal functions. It should be remarked that this proof does not require the general result of the existence of a mapping on a canonical hyperbolic slit domain. In the limit with $\zeta_1 = \zeta_2$ the analogous result as above holds, the situation being even more simple. It should be pointed out that Golusin obtained a slightly more general result, but (3) is the form which has been used in all applications so far. Also let us note that other of Golusin's inequalities, notably (17) in [1], can be obtained in a similar simple manner.

**2.** Now we would like to show how Golusin's inequality (1) can be used to simplify the proof of a result of Szegö [10]: Let the function $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n + \cdots$ be regular and univalent in $|z| < 1$. Then all the sections $c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$ $(n = 1, 2, \cdots)$ are univalent in the circle $|z| < 1/4$.

Szegö's proof uses a lower bound for the left hand side of the inequality (1) obtained from the distortion theorem. The general method works for $n \geqq 5$ with $n = 2, 3, 4$ requiring special devices, the case $n = 2$ being extremely simple. By the use of inequality (1) the case $n = 4$ is brought under the general method but it seems the case $n = 3$ must be treated by the method which Szegö used, namely by the use of Löwner's expressions for the coefficients $c_n$; [10], p. 196.

If $c_1 = 1$, then $|c_n| < en$, $|c_3| < 8$; cf. [10].

To prove the result for $n \geqq 4$ it is enough to prove the inequality

$$(4) \qquad |f(z_1) - f(z_2)|/|z_1 - z_2| > \left|\sum_{\nu=n+1}^{\infty} c_\nu (z_1^\nu - z_2^\nu)/(z_1 - z_2)\right|$$

for $|z_1| = |z_2| = 1/4$, $n \geqq 4$. By (1) and the distortion theorem, the quotient on the left of this inequality is at least

$$(1-r^2)/r^2 \cdot r^2/(1+r)^4 = (1-r)/(1+r)^3 = 48/125 \qquad (r = 1/4),$$

whereas the expression on the right does not exceed (cf. [10])

$$\frac{40}{256} + \frac{9.25 + 24.5 + 20}{27.256}\, e < \frac{40}{256} + \frac{41}{256} < \frac{48}{125}.$$

This completes the proof.

**3.** The problem of univalency of the sections of $f(z)$ was taken up by V. Levin [8] who showed that the $n$-th section was univalent for $|z| < 1 - 6 \log n/n$ $(n \geqq 17)$, an improvement of Szegö's result when $n$ is large. The preceding approach enables us to give a considerably better estimate. We may take $f(z) = z + c_2 z^2 + \cdots$, $f_n(z) = z + c_2 z^2 + \cdots + c_n z^n$. Again we use the fact that $f_n(z)$ will be univalent for $|z| \leqq r_n$ provided inequality (4) holds with $|z_1| = |z_2| = r_n$. This surely will be the case if the right hand side of this inequality is less than $(1-r_n)(1+r_n)^{-3}$. For this right hand side we have the upper estimate $\sum_{\nu=n+1}^{\infty} |c_\nu| \nu r_n^{\nu-1}$. To sharpen our final result somewhat we will now use Golusin's estimate $|c_\nu| < \frac{3}{4}e$, [3]. We then obtain above the further bound,

$$\tfrac{3}{4} e r_n^n (1-r_n)^{-3}\{n^2(1-r_n)^2 - (2n-1)r_n + 2n + 1\}.$$

Thus we will be assured of the desired inequality provided

$$r_n^n (1-r_n)^{-4}\{n^2(1-r_n)^2 - (2n-1)r_n + 2n + 1\} < (6e)^{-1}.$$

Substituting $r_n = 1 - \alpha/n$ and reducing, we see that this will be the case provided

$$n^4 e^{-\alpha}(\alpha^2 + 2\alpha + 2)\alpha^{-6} < (6e)^{-1}.$$

In order that this be satisfied we may take $\alpha = \beta_n \equiv 4 \log n - \log(4 \log n)$ if $n$ is sufficiently large.

The result is then that the $n$-th section is univalent for $|z| \leqq 1 - \beta_n/n$. Our result is meaningful only if the right hand side is positive, i. e., for $n$ large enough. Simple calculations show that both conditions on $n$ are satisfied provided $n \geqq 8$. For large $n$ the dominating term in our estimate is $1 - 4 \log n/n$, replacing Levin's $1 - 6 \log n/n$. Let us remark that this could not be improved beyond $1 - 2 \log n/n$, as was shown by Robertson [9].

**4.** The method of § **2** can, in addition, be used to establish a conjecture of Joh [6]. The result we shall prove is as follows:

Let the function $f(z) = z + a_3z^3 + a_5z^5 + \cdots$ be regular, univalent and odd for $|z| < 1$. Then all the sections $z + a_3z^3 + \cdots + a_{2n+1}z^{2n+1}$ $(n = 1, 2, \cdots)$ are univalent in the circle $|z| < 3^{-\frac{1}{2}}$ and the number $3^{-\frac{1}{2}}$ cannot be replaced by any larger one.

Joh proved this result in the case where all coefficients are real and in general for all values of $n$ but $n = 3$. The values $n = 1, 2$ are treated as special cases, the former being trivial, the latter requiring the use of Löwner's expressions for the coefficients. We now give a general proof for the cases $n \geqq 3$. For this purpose we must recall the bounds $|a_9| < 1.38$, $|a_{11}| < 1.70$, $|a_{2n+1}| < 17/5$; cf. [7].

From the inequality (1) and the known bounds for the modulus of an odd univalent function we derive

$$|f(z_1) - f(z_2)|/|z_1 - z_2| > (1 - r^2)r^{-2}r^2(1 + r^2)^{-2} = 3/8;$$
$$|z_1| = |z_2| = r = 3^{-\frac{1}{2}}.$$

On the other hand,

$$\sum_{\nu=4}^{\infty} |a_{2\nu+1}| \, |z_1^{2\nu+1} - z_2^{2\nu+1}|/|z_1 - z_2|$$
$$< (1.38)(1/9) + (1.70)(11/243) + (17/5)\sum_{\nu=6}^{\infty}(2\nu + 1)3^{-\nu}.$$

This last number is less than .331 which in turn is less than 3/8 $(= .375)$.

Now, as in § **2**, we see that all the sections for $n \geqq 3$ are univalent in $|z| < 3^{-\frac{1}{2}}$. That this bound is best possible is shown in [5].

The present method can also be used to simplify the proof of Joh's results on functions with bounded coefficients [6].

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# ON NETS OF CURVES ON NON-ORIENTABLE SURFACES.*

By S. K. Zaremba.

**1. Introduction.** The purely topological investigation of general nets of curves on closed surfaces was initiated by H. Hamburger [1] and B. v. Kerékjártó [2]. A further contribution by myself [4] was specifically concerned with orientable surfaces. However, nets of curves on non-orientable surfaces may also deserve attention, and their study is the object of the present paper.

We shall retain my former [4] definition of a *regular net of curves*, which is here re-stated for the reader's convenience:

1.1. Definition. $S$ being any closed surface (*i.e.*, any closed two-dimensional variety), a set $\mathfrak{R}$ of curves on $S$ will be called a *regular net of* curves on $S$, if to any point $M$ on $S$ (apart from a finite number of points, called *singular* with respect to $\mathfrak{R}$) there corresponds a neighborhood $U(M)$ of this point—called a *regular neighborhood*—capable of being topologically mapped on a neighborhood $V$ of the Euclidean plane, so that the set of all the arcs of curves of $\mathfrak{R}$ contained (with the possible exception of their ends) in $U(M)$ be transformed in the set of all rectilinear segments parallel to either of two distinct fixed directions of the aforesaid plane and contained in $V$ with the possible exception of their ends.

In what follows, $\mathfrak{R}$ will always denote a regular net of curves on a closed surface $S$. Obviously, it will not impair the generality of our reasoning if we limit ourselves to regular neighborhoods $U(M)$ bounded by closed curves.

1.2. Definition. According to the preceding definition, the arcs of curves of $\mathfrak{R}$ contained, possibly with the exception of their ends, in any regular neighborhood $U(M)$ can be so divided into two families that segments in the corresponding plane neighborhood $V$ representing arcs of the same family be parallel to each other; such a partition will be called *normal*. If the whole net $\mathfrak{R}$ can be so decomposed into two families of curves that the resulting partition of all the arcs of curves of $\mathfrak{R}$ contained, with the possible exception of their ends, in any regular neighborhood be normal, $\mathfrak{R}$ will be called *monodrome* and the aforesaid non-local decomposition of $\mathfrak{R}$ into two families of curves will again be called *normal*.

* Received July 24, 1950.

This definition too repeats essentially the corresponding definitions contained in [3] and [4]. The same applies to the following definition, which, however, is now so formulated as to apply to non-monodrome as well as to monodrome nets of curves, which is necessary in the case of non-orientable surfaces.

1.3. DEFINITION. If all the curves of a regular net $\mathcal{R}$ can be so oriented that in the plane neighborhood $V$ corresponding to any regular neighborhood $U(M)$ the resulting orientation of the segments parallel to one direction be coherent, $\mathcal{R}$ will be called *bilateral*. Similarly, in the case of a monodrome net, if the curves of one of its constituent families (corresponding to a normal partition of $\mathcal{R}$) can be so oriented that in the plane neighborhood corresponding to any regular neighborhood $U(M)$ the resulting orientation of the rectilinear images of the arcs of curves of this family be coherent, the aforesaid family of curves will be called *bilateral*.

Whereas on an orientable surface, in the case of a monodrome net of curves, either both constituent families are bilateral or both not, on the contrary, on a non-orientable surface two families of curves obtained by a normal partition of a monodrome net cannot both be bilateral, but there are examples of monodrome nets of curves in which one of the two constituent families is bilateral. On the other hand, we shall see that (again in contrast with orientable surfaces, on which only monodrome nets of curves can be bilateral), on non-orientable surfaces non-monodrome nets are the only ones capable of being bilateral.

The contributions of H. Hamburger [1] and B. v. Kerékjártó [2] to the theory of nets of curves on surfaces were mainly based on a concept of index related to that of Kronecker and defined for a special kind of curvilinear polygons, each of which was supposed to be contained in one neighborhood. The idea of a non-local index, as introduced in [4], depends intrinsically on the orientation of the surface, so that it cannot be applied to the case we are now investigating. However, on non-orientable surfaces, it is possible to introduce in a similar way an index mod $\frac{1}{2}$ and, in the case of monodrome sets, even an index mod 1, which is sufficient to obtain, in particular, criteria of monodromy and bilaterality analogous to those previously found for nets of curves on orientable surfaces.

## 2. The index mod $\frac{1}{2}$.

We shall need the following definition:

2.1. DEFINITION. By a *normal polygon* of the regular net of curves $\mathcal{R}$ we mean any closed curvilinear polygon without multiple points, whose sides are arcs of curves of $\mathcal{R}$, two consecutive sides being always contained in a common regular neighborhood $U(M)$ of their common vertex $M$.

Having chosen $U(M)$ and having carried out a normal partition of the arcs of curves of $\mathfrak{R}$ contained, with the possible exception of their ends, in $U(M)$, we can determine whether the two sides of the given polygon meeting at $M$ belong to the same family of curves or not (evidently, the result does not depend on the particular choice of $U(M)$); in the first case, $M$ will be called a *neutral* vertex of the polygon.

In the second case, supposing the neighborhood $U(M)$, as well as the polygon itself, to be oriented, the vertex $M$ can be labeled as positive or negative in the following way: We extend the two sides of the polygon adjacent to $M$ as far as the boundary of $U(M)$ and so as to be still in the presence of arcs of curves of $\mathfrak{R}$. Let $LM$ and $MN$ be these extended sides, their orientations from $L$ to $M$ and from $M$ to $N$ being consistent with that of the given polygon. Consider now that arc $LN$ of the boundary of $U(M)$ which does not meet the extensions of $LM$ and $NM$ beyond $M$. If the orientation of this arc from $L$ to $N$ agrees with that induced by the orientation of $U(M)$, $M$ will be reckoned as a *positive vertex*; otherwise $M$ will be called *negative*.

In the case of an orientable surface, its orientation could be given, allowing to deduce therefrom the orientation of every $U(M)$, and the index of an oriented normal polygon $a$ was defined by the formula

$$\text{ind}\,(a) = 1 - \tfrac{1}{4}s + \tfrac{1}{4}r, \tag{1}$$

where $s$ was the number of its positive vertices and $r$ that of its negative vertices [4]. The index of a normal polygon bounding a neighborhood of a singular point coincided with the index (according to Kerékjártó [2]) of this singular point whenever the orientation of the polygon was that induced by the orientation of the neighborhood.

If the surface is not orientable, this definition of the non-local index (as opposed to that of the index of a singular point, which is of local character) cannot be applied directly. We can, however, adopt an arbitrary orientation of a neighborhood of a starting point on a given normal polygon and propagate this orientation while describing the polygon in a given direction, thus giving a meaning to the index defined by (1). Obviously, if we return to the starting point with the opposite orientation, *i. e.*, if the cut on the surface corresponding to the given polygon is one-sided, the value found for the index depends on the starting point. More exactly, it is easy to see that, if the point moves so as to pass through a non-neutral vertex of the polygon, this point changes from positive to negative or conversely, so that the change in the value obtained for the index amounts to $\pm\frac{1}{2}$. Thus in the general case the index, whose value may be any multiple of $\frac{1}{4}$, is deter-

mined only mod $\frac{1}{2}$. It can therefore be defined in a simpler way as being equal, mod $\frac{1}{2}$, to one-quarter of the number of non-neutral vertices of the given polygon. This amounts to a definition of the values of the index as residue classes modulo one-half. Evidently, the value of the index of a normal polygon, as defined above, does not depend on its orientation; this is why, in what follows, topological complexes will be considered only mod 2. If $a$ is any normal polygon, its index modulo one-half will be denoted by $\mathrm{ind}_{\frac{1}{2}}(a)$.

It follows from our definition that if the normal polygon $a$ bounds a neighborhood containing exactly one singular point, let us say $P$, the residue class mod $\frac{1}{2}$ of the index of $P$ as defined by Kerékjártó [2] is $\mathrm{ind}_{\frac{1}{2}}(a)$. In particular, the index mod $\frac{1}{2}$ of a normal polygon bounding a neighborhood without singular points is zero.

2.2. THEOREM. *Let $a$ and $b$ be two normal polygons whose common part forms an open polygon (which can always be assumed to be bounded by two vertices common to $a$ and $b$), so that their sum* mod 2 *(in the sense of the theory of topological complexes), which will be denoted by $a+b$, be another normal polygon. Then* $\mathrm{ind}_{\frac{1}{2}}(a) + \mathrm{ind}_{\frac{1}{2}}(b) = \mathrm{ind}_{\frac{1}{2}}(a+b)$.

*Proof.* The contribution of any vertex belonging to one only of the polygons $a$ and $b$ is the same for both sides of this equality. The contributions of any interior vertex of the common part of $a$ and $b$ to $\mathrm{ind}_{\frac{1}{2}}(a)$ and to $\mathrm{ind}_{\frac{1}{2}}(b)$ are the same, and since they amount to 0 or $\frac{1}{2}$, they cancel each other modulo $\frac{1}{2}$ in $\mathrm{ind}_{\frac{1}{2}}(a) + \mathrm{ind}_{\frac{1}{2}}(b)$, whereas in $a+b$ the same vertex does not appear at all. Finally, any of the two ends of the common part of $a$ and $b$ must obviously be a non-neutral vertex either in both polygons $a$ and $b$ or in one of them only; in the former case, and only in that case, such a vertex is neutral in $a+b$, so that in both cases its contributions to $\mathrm{ind}_{\frac{1}{2}}(a) + \mathrm{ind}_{\frac{1}{2}}(b)$ and to $\mathrm{ind}_{\frac{1}{2}}(a+b)$ are congruent mod $\frac{1}{2}$. Thus adding together mod $\frac{1}{2}$ all these contributions to $\mathrm{ind}_{\frac{1}{2}}(a) + \mathrm{ind}_{\frac{1}{2}}(b)$ on the one hand and to $\mathrm{ind}_{\frac{1}{2}}(a+b)$ on the other, we find the same residue classes. Hence the proof is complete.

By modifying slightly the argument in [4], § 2.5, to suit topological complexes mod 2, we obtain the two following lemmas:

2.3. LEMMA. *If $k$ is any component of the common part of two normal polygons, $c_1$ and $c_2$, and is bounded by two distinct vertices common to $c_1$ and $c_2$, the sum $c_1 + c_2$, treated as a complex* mod 2, *can be resolved into a finite number of normal polygons, say $a_1, \cdots, a_n$, composed of sides of $c_1$ and $c_2$ with the exclusion of those belonging to $k$, so that no side occurs in more than two of the polygons $a_1, \cdots, a_n$.*

2.4. LEMMA. *On the assumption of the preceding lemma,*

$$\operatorname{ind}_{\frac{1}{2}}(c_1) + \operatorname{ind}_{\frac{1}{2}}(c_2) = \sum_{i=1}^{i=n} \operatorname{ind}_{\frac{1}{2}}(a_i).$$

Hence, by mathematical induction with respect to the total number of curvilinear segments involved, we deduce the following theorem.

2.5. THEOREM. *If $c_1, \cdots, c_m$ are normal polygons whose sum* mod 2 *is the zero complex,*

$$\sum_{i=1}^{i=n} \operatorname{ind}_{\frac{1}{2}}(c_i) = 0.$$

2.6. COROLLARY. *If $a_1, \cdots, a_m$ and $b_1, \cdots, b_n$ are normal polygons such that the complexes* mod 2 *denoted by $a_1 + \cdots + a_m$ and $b_1 + \cdots + b_n$ respectively are equal,*

$$\sum_{i=1}^{i=m} \operatorname{ind}_{\frac{1}{2}}(a_i) = \sum_{i=1}^{i=n} \operatorname{ind}_{\frac{1}{2}}(b_i).$$

Finally, the following theorem, corresponding to 2 · 9 in [4], can be stated and proved in a simplified way owing to the replacement of ordinary integers by residue classes:

2.7. THEOREM. *If $s$ is the remainder class* mod $\frac{1}{2}$ *of the sum of the indices of the singular points contained in a part of the surface $S$ bounded by the normal polygons $a_1, \cdots, a_m$,*

$$s = \sum_{i=1}^{i=m} \operatorname{ind}_{\frac{1}{2}}(a_i).$$

*Proof.* This part of $S$ can be decomposed into a finite number of neighborhoods, none of which contains more than one singular point, and which are bounded by some normal polygons, say $b_1, \cdots, b_n$. Clearly

$$s = \sum_{i=1}^{i=n} \operatorname{ind}_{\frac{1}{2}}(b_i).$$

On the other hand, $a_1 + \cdots + a_m = b_1 + \cdots + b_n$ and therefore, by 2.6,

$$s = \sum_{i=1}^{i=n} \operatorname{ind}_{\frac{1}{2}}(b_i) = \sum_{i=1}^{i=m} \operatorname{ind}_{\frac{1}{2}}(a_i).$$

Now criteria for the monodromy (see 1.2) of regular nets of curves can readily be obtained.

2.8. THEOREM. *$\mathfrak{R}$ is monodrome, if, and only if, the indices* mod $\frac{1}{2}$ *of all the normal polygons on $S$ are* 0.

This theorem is the counterpart of the result of 2.31 in [3] and of 3.1 in [4].

*Proof.* Wishing to divide $\mathcal{R}$ into two families, say $\mathcal{A}$ and $\mathcal{B}$, let us start from any regular neighborhood $U(M)$, where the arcs of the curves of $\mathcal{R}$ have undergone a normal partition into two families, $\mathcal{A}$ and $\mathcal{B}$, and let us propagate this partition similarly to the analytic continuation of functions of complex variables. Obviously, $\mathcal{R}$ is monodrome if, and only if, we always come back to $U(M)$ with the initial determination of $\mathcal{A}$ and $\mathcal{B}$, and the path followed can evidently be supposed to be a normal polygon. Now, when tracing a normal polygon, we meet non-neutral vertices when, and only when, we pass from an arc of a curve $\mathcal{A}$ to an arc of a curve of $\mathcal{B}$ or conversely, so that we return with the initial partition of $\mathcal{R}$ if and only if the number of non-neutral vertices is even, *i. e.*, if and only if the index mod $\frac{1}{2}$ is 0.

As it may be preferable not to have to consider all possible normal polygons, let us finally replace the preceding proposition by a more special criterion, directly deducible from 2. 7 and 2. 8.

2. 9. THEOREM. *A regular net of curves $\mathcal{R}$ on a surface $S$ is monodrome if and only if the indices* mod $\frac{1}{2}$ *of the normal polygons of a set forming a base of homology for $S$, as well as the residues* mod $\frac{1}{2}$ *of the indices of the singular points are all* 0.

**3. The index mod 1.** Throughout this section, we assume the monodromy of $\mathcal{R}$, and denote by $\mathcal{A}$ and $\mathcal{B}$ the two families of curves obtained as a result of a normal decomposition of $\mathcal{R}$.

3. 1. DEFINITION. On this assumption, if a normal polygon $a$ is given, and if we apply the procedure described in 2. 1, starting from any interior point of a side $a$ belonging to $\mathcal{A}$, the result thus obtained will be determined mod 1. Indeed, if the starting point remains on the same side of $a$, nothing will be changed in the computation of the index. If, however, this point moves to the next side of $a$ belonging to $\mathcal{A}$, it has to pass through two non-neutral vertices. The contribution of each of them to the index changes by $\pm \frac{1}{2}$ if the cut determined by $a$ is one-sided, and does not change at all if this cut is two-sided; so that the value obtained for the index is either unchanged or changed by $\pm 1$, leaving the corresponding residue class mod 1 unchanged in any case. Hence, if the starting point moves from one side of $a$ belonging to $\mathcal{A}$ to any other side also belonging to $\mathcal{A}$, this residue class will remain the same. It will be called the *index* mod 1 *of $a$ with respect to* $\mathcal{A}$ and will be denoted by $\operatorname{ind}_{\mathcal{A}}(a)$.

According to 2. 8, the only possible values for $\operatorname{ind}_{\mathcal{A}}(a)$ are the residue classes of 0 and $\frac{1}{2}$, so that $-\operatorname{ind}_{\mathcal{A}}(a) = \operatorname{ind}_{\mathcal{A}}(a)$. Hence the index mod 1 depends neither on the orientation of $a$ nor on that of the initial neighborhood.

3. 2. *Remark.* On the other hand, if the cut determined by $a$ is one-sided, $\text{ind}_{\mathfrak{B}}(a) = \text{ind}_{\mathfrak{A}}(a) + \frac{1}{2}$. This property of the index mod 1 is in contrast with the corresponding properties both of the index mod $\frac{1}{2}$ and of the index defined in [4] for orientable surfaces.

3. 3. THEOREM. *If $a$ and $b$ are two normal polygons whose common part is an open polygon bounded by two vertices common to $a$ and $b$, so that $a + b$ is another normal polygon,*

$$\text{ind}_{\mathfrak{A}}(a) + \text{ind}_{\mathfrak{A}}(b) = \text{ind}_{\mathfrak{A}}(a + b).$$

*Proof.* The polygons $a$ and $b$ can be supposed to be so oriented that the resulting orientations of their common part be opposite. The orientations of the regular neighborhoods covering together both polygons can so be determined that they coincide on the common part of $a$ and $b$. On these assumptions, it is easy to see that each of the two ends of the common part of $a$ and $b$ either is non-neutral in both $a$ and $b$ and has the same sign in $a$ as in $b$, but becomes neutral in $a + b$, or is non-neutral in one only of the polygons $a$ and $b$ and remains so in $a + b$, but with the opposite sign. Thus the contribution of each of these two vertices to $\text{ind}_{\mathfrak{A}}(a + b)$ differs from that to $\text{ind}_{\mathfrak{A}}(a) + \text{ind}_{\mathfrak{A}}(b)$ by $\pm \frac{1}{2}$ and both these differences, added together mod 1, cancel each other. On the other hand, the contributions to $\text{ind}_{\mathfrak{A}}(a) + \text{ind}_{\mathfrak{A}}(b)$ of any other vertex common to $a$ and $b$ cancel each other, while the vertex itself disappears in $a + b$. Finally, each of the vertices occurring in one only of the polygons $a$ and $b$ yields the same contribution to both sides of the equality to be proved. Hence $\text{ind}_{\mathfrak{A}}(a) + \text{ind}_{\mathfrak{A}}(b) = \text{ind}_{\mathfrak{A}}(a + b)$.

The preceding theorem can be generalized as follows:

3. 4. LEMMA. *On the assumptions of Lemma 2. 3, if $\mathfrak{R}$ is monodrome and split up into two families of curves, $\mathfrak{A}$ and $\mathfrak{B}$ (this partition being normal),*

$$\text{ind}_{\mathfrak{A}}(c_1) + \text{ind}_{\mathfrak{A}}(c_2) = \sum_{i=1}^{i=n} \text{ind}_{\mathfrak{A}}(a_i).$$

Hence, by mathematical induction with respect to the total number of curvilinear segments involved, we deduce the following theorem:

3. 5. THEOREM. *If $c_1, \cdots, c_m$ are normal polygons whose sum* mod 2 *is the zero complex,*

$$\sum_{i=1}^{i=m} \text{ind}_{\mathfrak{A}}(c_i) = 0.$$

3. 6. COROLLARY. *If $a_1, \cdots, a_m$ and $b_1, \cdots, b_n$ are normal polygons such that the complexes* mod 2 *represented by $a_1 + \cdots + a_m$ and $b_1 + \cdots + b_n$ are equal,*

$$\sum_{i=1}^{i=m} \mathrm{ind}_{\mathcal{A}}(a_i) = \sum_{i=1}^{i=n} \mathrm{ind}_{\mathcal{A}}(b_i).$$

Corresponding to 2. 7, the following can be proved.

3. 7. THEOREM. *If $s$ is the residue class* mod 1 *of the sum of the indices of the singular points contained in a part of $S$ bounded by the normal polygons $a_1, \cdots, a_m$,*

$$s = \sum_{i=1}^{i=m} \mathrm{ind}_{\mathcal{A}}(a_i).$$

The following criteria for the bilaterality (see 1. 3) of a family of curves are analogous to those for the monodromy of a net of curves (see § 2).

3. 8. THEOREM. *The family $\mathcal{A}$ is bilateral if and only if the indices* mod 1 *with respect to $\mathcal{A}$ of all the normal polygons on $S$ are* 0.

*Proof.* The proof is similar to that in [4], § 3. 5.

In order to give an orientation to the curves of $\mathcal{A}$, let us start from any normal neighborhood $U(M)$, where the arcs of the curves of $\mathcal{A}$ are oriented in a coherent way and propagate this partition so that, in the part common to two overlapping neighborhoods, the corresponding orientations of the curves of $\mathcal{A}$ agree with each other.

$\mathcal{A}$ is bilateral if and only if we always return to the initial neighborhood with the same orientation of the curves of $\mathcal{A}$, whenever we describe a normal polygon. In agreement with 3. 1, the starting point can always be supposed to be on a side of the polygon formed by an arc of a curve of $\mathcal{A}$. Such a side will be called an $\mathcal{A}$-side and will be regarded as *positive* if its orientation derived from that of the family agrees with that of the polygon; otherwise it will be regarded as *negative*. If two consecutive $\mathcal{A}$-sides have the same sign, they are separated either by a neutral vertex or by two vertices of opposite signs, possibly together with an unspecified number of neutral vertices; in both cases, the total contribution of these vertices to the index of the polygon is 0. On the other hand, it is easy to see that two consecutive $\mathcal{A}$-sides of opposite signs are separated by two vertices of the same sign, possibly together with an unspecified number of neutral vertices, giving together a contribution to the index of the polygon amounting to $\pm \frac{1}{2}$.

After having traced the polygon, we come back to the starting point with the original orientation of the curves of $\mathcal{A}$ if and only if we have passed as many times from a positive $\mathcal{A}$-side to a negative side as from a negative side to a positive one; that is to say (according to what we have just seen), if and only if the value found for the index following the definition 3. 1 is an integer. Hence the proof is complete.

As a direct consequence of the preceding theorems, we find the following criterion of bilaterality:

3. 9. THEOREM. *The family $\mathcal{A}$ is bilateral if and only if the indices* mod 1 *with respect to $\mathcal{A}$ of a set of normal polygons forming a base of homology for $S$, as well as the residues* mod 1 *of the indices of the singular points are all* 0.

**4. Bilateral nets of curves.** This paper deals mainly with nets of curves on non-orientable surfaces, but the propositions established above are valid for orientable surfaces as well. In order to show the contrast between the cases of orientable and non-orientable surfaces, we shall first deduce from these propositions the following theorem:

4. 1. THEOREM. *A bilateral net of curves on an orientable surface is necessarily monodrome.*

*Proof.* Let $a$ be a normal polygon with respect to a bilateral net, $\mathcal{R}$, of curves on an orientable surface $S$. We denote by $M$ any point on $a$ other than a vertex and we choose an orientation of $S$ arbitrarily. Starting from $M$, let us follow $a$ in any of the two directions, counting on our way, according to formula (1), the contributions of the various vertices to the index of $a$ and propagating, at the same time, from neighborhood to neighborhood, a normal partition of $\mathcal{R}$ into two families of curves, say $\mathcal{A}$ and $\mathcal{B}$, as well as an orientation of $\mathcal{R}$, both initially determined in a regular neighborhood $U(M)$ of $M$. Returning to $M$ for the first time, we may bring with us the initial determination of $\mathcal{A}$ and $\mathcal{B}$, or else the two families can be permuted. However, if we continue following $a$ in the same direction and propagating both the orientation of $\mathcal{R}$ and its partition into $\mathcal{A}$ and $\mathcal{B}$, the next time we reach $M$ again, the determination of $\mathcal{A}$ and $\mathcal{B}$ will be the initial one. $\mathcal{R}$ being bilateral, $\mathcal{A}$ and $\mathcal{B}$ will also be oriented as before. Hence, by an argument similar to that of 3. 8, we see that if we keep counting on our way the contributions of the encountered vertices to the index of $a$, the total with which we reach $M$ after two circuits must be an integer. Since, owing to the orientability of $S$, each vertex appears both times with the same sign, this total is twice the sum obtained after one circuit. This sum is therefore a multiple of $\frac{1}{2}$, which means that $\mathrm{ind}_{\frac{1}{2}}(a) = 0$. The normal polygon $a$ being arbitrary, this implies, according to 2. 8, that $\mathcal{R}$ is monodrome.

4. 2. It is also clear that if a monodrome net of curves on an orientable surface is normally decomposed into two families, both these families are simultaneously bilateral or not.

The converse of 4. 1 and 4. 2 is true in the case of non-orientable surfaces.

Indeed, since on a non-orientable surface covered by a regular net of curves there are always normal polygons determining one-sided cuts, 3. 2 and 3. 8 imply the following proposition.

4. 3. PROPOSITION. *If a monodrome net of curves on a non-orientable surface is subject to a normal decomposition into two families of curves, both of them cannot be bilateral.*

This can also be deduced from the fact that a coherent orientation of both these families of curves would define a coherent orientation of the surface.

4. 4. However, one of the two families of curves obtained by the normal decomposition of a net on a non-orientable surface can be bilateral. A simple example of such a net on the projective plane is afforded by a pencil of concentric circles supplemented by the straight line at infinity, the other family consisting of the diameters common to all the circles of the pencil. The possibility of imparting a coherent orientation to the first family is obvious, and it is not difficult to see that the family composed of the diameters is not bilateral.

4. 5. On the other hand, non-monodrome nets of curves on non-orientable surfaces can be bilateral. This is shown by the following example:

We obtain a Klein bottle from the square $Q$ of the $(x, y)$-plane defined by the inequalities $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$, if we identify the points $(x, 0)$ and $(x, 2\pi)$ for any value of $x$, and the points $(0, y)$ and $(2\pi, 2\pi - y)$ for any value of $y$. On this surface, the characteristics of the differential equation

$$\sin \tfrac{1}{2}x\, dx^2 + 2 \cos \tfrac{1}{2}x\, dxdy - \sin \tfrac{1}{2}x\, dy^2 = 0$$

or

$$(\cos \tfrac{1}{4}x\, dx - \sin \tfrac{1}{4}x\, dy)(\sin \tfrac{1}{4}x\, dx + \cos \tfrac{1}{4}x\, dy) = 0 \tag{2}$$

form a regular net of curves without singular points.

The fact that this net is bilateral (and therefore also not monodrome) can best be seen as follows: The two orthogonal vectors $u$ and $v$ of coordinates $(\sin \frac{1}{4}x, \cos \frac{1}{4}x)$ and $(\cos \frac{1}{4}x, -\sin \frac{1}{4}x)$ respectively are tangent to the characteristics of (2) at any point of $Q$ and locally they determine an orientation of the curves of the net. Since they are continuous, in order to prove that this orientation is coherent, we have only to verify that the orientations of the curves determined by these vectors agree on both pairs of sides of $Q$ which are identified. But this is clear as far as the sides $y = 0$ and $y = 2\pi$ are concerned, the identified points having the same $x$-value, which alone determines the vectors. On the side $x = 0$, the vectors $u$ and $v$ become $(0, 1)$ and $(1, 0)$, whereas on the side $x = 2\pi$ they become $(1, 0)$ and $(0, -1)$,

respectively. Owing to the manner of identifying these two sides, the vector $(0, 1)$ on $x = 0$ coincides with $(0, -1)$ on $x = 2\pi$, so that the two vectors $u$ and $v$ are only subjected to a permutation when we cross the meridian corresponding to the two sides $x = 0$ and $x = 2\pi$ of $Q$. This permutation shows that the net is not monodrome, and the fact that otherwise the directions of the pairs of vectors agree with each other proves that the net is bilateral, one of its two possible orientations being determined by the couple of vectors $u$ and $v$.

It should be noted that in general the definition of the index by means of formula (1), such as it was stated in [4] and recalled above (cf. 2. 1), can be applied, in the case of a non-orientable surface, not only to normal polygons bounding neighborhoods (for instance, those of singular points), but also to any normal polygons corresponding to two-sided cuts on the given surface. The value of the index then depends on the orientation of the polygon and on that of the surface in its neighborhood, but the question of its being an integer or not is clearly independent of both these orientations (see also [4], § 2. 2). This leads to conclude with the following criterion of bilaterality for regular nets of curves, which applies to orientable and non-orientable surfaces alike and includes, as a particular case, a previously stated criterion of bilaterality for nets of curves on orientable surfaces (cf. [4], § 3. 5).

4. 6. THEOREM. *A regular net of curves on a surface $S$ is bilateral if and only if the indices of normal polygons corresponding to two-sided cuts of $S$ are all integers, whereas the indices* mod $\frac{1}{2}$ *of normal polgons corresponding to one-sided cuts of $S$ are all equal to* $\frac{1}{4}$.

*Proof.* (i) The condition is necessary. With respect to polygons corresponding to two-sided cuts of $S$, the arguments of 4. 1 and 3. 8 can be applied to show that the indices must all be integers. On the other hand, if a normal polygon corresponding to a one-sided cut of $S$ had an index mod $\frac{1}{2}$ equal to 0, then, by the argument of 2. 8, the net of curves could be split up normally into two families of curves in a neighborhood of the polygon; so that the definition 3. 1 of the index mod 1 could be applied to each of these two families. But according to 3. 2, the indices mod 1 of the polygon with respect to the two families cannot vanish simultaneously. Consequently, at least one of these two families must fail to be bilateral, which precludes the bilaterality of the whole net.

(ii) The condition is sufficient. In an arbitrarily chosen regular neighborhood $U(M)$, let us split up normally the given net of curves into two families and determine the orientation of one of these two families, which

will be denoted by $\mathcal{A}$. The determination of the family $\mathcal{A}$, as well as its orientation, can be propagated from neighborhood to neighborhood through their overlapping parts. The case of an orientable surface having been dealt with previously, we can suppose $S$ to be non-orientable, so that there exist normal polygons corresponding to one-sided cuts. According to the argument of 2. 8, if, following such a polygon, we propagate the determination of $\mathcal{A}$ together with its orientation, we shall return to $U(M)$ with a changed determination of the family $\mathcal{A}$. This will precisely give the orientation of the second family of curves in $U(M)$, thus enabling us to orient the whole net of curves, provided that we are not led to inconsistencies.

The consistency of the aforesaid way of determining the orientation of the given net of curves will be proved if we show that any closed path on the surface has either of the two following properties:

($\alpha$) if we follow it, starting from any neighborhood where the family $\mathcal{A}$, as well as its orientation, is locally determined, we come back with the initial determination and the initial orientation of $\mathcal{A}$;

($\beta$) if we trace it once, starting from any point of a neighborhood where both the family $\mathcal{A}$ and its orientation are locally given, we return with a different determination of $\mathcal{A}$, but if we trace the same path once more in the same direction, eventually we arrive at the starting point with the initial determination and the initial orientation of $\mathcal{A}$.

Obviously, we can always suppose the path to be formed by a regular polygon of the net of curves. If it corresponds to a two-sided cut of the surface, its index is an integer, and the argument of 3. 8 shows that the polygon has the property ($\alpha$).

If, on the other hand, the polygon corresponds to a one-sided cut of the surface, its index mod $\frac{1}{2}$ is $\frac{1}{4}$. Consequently, as is shown by the proof of 2. 8, after one tour, we reach the initial point with a changed determination of $\mathcal{A}$, *i. e.*, the two families into which the net is locally split up are permuted. Therefore, after having traced the polygon once more in the same direction, we recover the initial determination of $\mathcal{A}$, and we have only to prove that we return with the initial orientation of $\mathcal{A}$.

We can always suppose the starting point to be on a side formed by an arc of a curve of $\mathcal{A}$. If an orientation of the initial neighborhood has been chosen, it can be propagated along the polygon with the determination and the orientation of $\mathcal{A}$; so that, while following the polygon, we can count the contributions of its vertices to the index, in accordance with formula (1). It is clearly possible to discard neutral vertices on the polygon, that is to say, to consider two sides connected by a neutral vertex as one side. Then, while

we follow the polygon, every other side is labeled as belonging to $\mathcal{A}$ and, as such, provided with an orientation. This orientation may coincide or not with that of the polygon. As in the proof of 3. 8, such a side will be called *positive* in the former case and *negative* in the latter. Between two consecutive sides considered as belonging to $\mathcal{A}$ there are two vertices and, as in 3. 8, it is easy to see that these vertices are of the same sign if and only if the aforesaid sides are of opposite signs. Hence the joint contribution to the index of two vertices situated between two sides attributed to $\mathcal{A}$ is 0 if these sides have the same sign and equals $\pm \frac{1}{2}$ if they are of opposite signs.

On the other hand, if we trace the polygon twice, we pass through each vertex twice, but (owing to the one-sidedness of the corresponding cut) with opposite orientations of its neighborhood, so that the same vertex appears once as positive and once as negative, and its total contribution to the index is 0. The total value for the index is therefore an integer (it is immaterial whether or not we count the constant term in (1)). This shows that we pass as many times from a positive side to a negative side as from a negative side to a positive one. In other terms, after describing the polygon twice in the same direction, we return with the initial orientation of the family $\mathcal{A}$, i. e., the polygon has the property $(\beta)$. Hence the proof is complete.

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# GENERALIZED STIELTJES-POST INVERSION FORMULA FOR INTEGRAL TRANSFORMS INVOLVING A PARAMETER.*

By L. C. Hsu.

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**1. Introduction.** The object of this paper is to establish an inversion formula for a kind of integral transforms involving a parameter. The classical Stieltjes-Post inversion formula for Laplace transform is included as a particular case. Our method is an extension of the method of Laplace, Haviland [3] and Widder.

Let $f(x)$ be the real Laplace transform of $g(u)$, namely,

$$f(x) = \int_0^\infty e^{-xu} g(u)\, du, \tag{1}$$

where the right-hand side of (1) converges for some $x$ and $g(u)\,\varepsilon\, L(o, R)$ for every positive $R$. It is a well-known result discovered by Stieltjes and Post and generalized by Widder (see [1; pp. 276-289], [2; pp. 723-781]) that, for every positive $t$ in the Lebesgue set for $g(u)$, $g(t)$ can be expressed in terms of the derivative $f^{(\lambda)}(\lambda/t)$ with $\lambda \to +\infty$, viz.

$$g(t) = \lim_{\lambda\to\infty} (-1)^\lambda (\lambda/t)^{\lambda+1} f^{(\lambda)}(\lambda/t)/\lambda!. \tag{2}$$

By Stirling's formula we may rewrite (2) as

$$g(t) = \lim_{\lambda\to\infty} (-1)^\lambda (2\lambda\pi)^{-\frac{1}{2}} (e/t)^\lambda (\lambda/t) f^{(\lambda)}(\lambda/t). \tag{2}^1$$

Since (1) can be differentiated any number of times (under the integral sign) within its region of convergence [1; p. 57], we have

$$(-1)^\lambda f^{(\lambda)}(\lambda/t) = \int_0^\infty e^{-\lambda u/t} u^\lambda g(u)\, du, \qquad (\lambda = 1, 2, 3, \cdots). \tag{3}$$

Hence (2) or $(2)^1$ may also be considered as an inversion formula for the integral transforms defined by (3).

Instead of the special transforms of (3), we now consider integral transforms of the form

$$f(\lambda, t) = \int_0^\infty \psi(u, \lambda, t) g(u)\, du, \tag{4}$$

where $\lambda$ is a real parameter and the right-hand side of (4) is assumed to be convergent for some $t > 0$, and $\psi(u, \lambda, t)$ is a real valued function of

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$u, \lambda, t (u \geq 0, \lambda > 0, t > 0)$ having positive values when $u, \lambda, t > 0$. As in the ordinary case we call $f$, $g$, $\psi$ the generating function, determining function and transform function respectively.

Under certain conditions we shall prove a general inversion formula of the form

$$(5) \quad g(\theta(t)) = \lim_{\lambda \to \infty} (\{-(\partial/\partial u)^2 \log \psi(u, \lambda, t)\}/\{2\pi(\psi(u, \lambda, t))^2\})^{\frac{1}{2}}_{u=\phi(\lambda,t)} f(\lambda, t),$$

where $\phi(\lambda, t)$ is a function satisfying the equation

$$(6) \quad [\partial/\partial u \log \psi(u, \lambda, t)]_{u=\phi(\lambda,t)} = 0$$

for all large values of $\lambda$ and certain $t > 0$, and

$$(7) \quad \lim_{\lambda \to \infty} \phi(\lambda, t) = \theta(t) > 0.$$

The relation (7) may be used for determining $\theta(t)$.

It is easy to verify that the Stieltjes-Post inversion formula $(2)^1$ is implied by (5). For if we take $\psi(u, \lambda, t) = e^{-\lambda u/t} u^\lambda$, then $\partial/\partial u \log \psi(u, \lambda, t) = \lambda(1/u - 1/t)$, $(\partial/\partial u)^2 \log \psi(u, \lambda, t) = -\lambda u^{-2}$, so that $u = t$ is the only solution satisfying (6), and consequently $\phi(\lambda, t) = \theta(t) = t$. Hence a direct substitution into (5) yields the results $(2)^1$.

**2. Some lemmas.** In order to investigate sufficient conditions under which the general formula (5) is valid, we need some lemmas concerning the asymptotic evaluation of integrals. In what follows we write $(\partial/\partial u)F(u, v) = F_u(u, v)$, $(\partial/\partial u)^2 F = F_{uu}$, etc. $A$ will be used to denote a certain positive number not necessarily the same in different occurrences.

LEMMA 1. *Let $F(u, v)$, $F_u(u, v)$, $F_{uu}(u, v)$ be continuous functions defined in the domain $D$ $(-\infty \leq a \leq u \leq b \leq \infty, v > 0)$ such that*

(i) *there is a continuous function $u = \phi(v)$ satisfying $F_u(\phi(v), v) \equiv 0$ for all large $v$, and $\phi(v) \to \xi$ $(a < \xi < b)$ as $v \to \infty$;*

(ii) *$F_{uu}(u, v) < 0$ for all $u$ $(a < u < b)$ and large $v$;*

(iii) *$F_{uu}(u, v) \sim F_{uu}(\xi, v) \to -\infty$ as $u \to \xi$, $v \to \infty$ simultaneously.*[1]

*Then as $\lambda \to \infty$ we have*

$$(8) \quad \int_a^b \exp[F(u, \lambda)]\, du \sim \exp[F(\phi(\lambda), \lambda)](-2\pi/F_{uu}(\xi, \lambda))^{\frac{1}{2}}.$$

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[1] Precisely speaking, $F_{uu}(u, v) \sim F_{uu}(\xi, v)$ means that for any given $\epsilon > 0$ there exist a small number $\delta = \delta(\epsilon) > 0$ and a large number $M = M(\epsilon)$ such that $|F_{uu}(u, v)/F_{uu}(\xi, v) - 1| < \epsilon$ whenever $|u - \xi| < \delta$, $v > M$. Similarly $F_{uu}(u, v) \to -\infty$ means that $-F_{uu}(u, v) > N$ for any given $N$, provided that $|u - \xi| < \delta$, $v > M$, where $\delta$, $M$ depend on $N$.

*Proof.* It suffices to give a proof for the case $a=-\infty$, $b=+\infty$. By hypotheses (i) and (ii) it is obvious that for every large $v=\lambda$ (say $\lambda>N$) the integrand function $\exp[F(u,\lambda)]$ always assumes a maximum at $u=\phi(\lambda)$, since $F_u(\phi(\lambda),\lambda)=0$ and $F_{uu}<0$. Moreover, hypotheses (i) and (ii) ensure that the implict function theorem is applicable to $F_u(u,v)=0$, so that $u=\phi(v)$ is actually a single-valued function of $v$.

Fix $\lambda$ and set

$$\int_{-\infty}^{+\infty}\exp[F(u,\lambda)]\mathrm{du}=\int_{-\infty}^{\phi(\lambda)-\epsilon(\lambda)}+\int_{\phi(\lambda)-\epsilon(\lambda)}^{\phi(\lambda)+\epsilon(\lambda)}+\int_{\phi(\lambda)+\epsilon(\lambda)}^{+\infty}=J_1+J_2+J_3,$$

where $\epsilon(\lambda)>0$ is a certain function tending to zero with $1/\lambda$. By Taylor's theorem with remainder we have

$$J_2(\lambda)\exp[-F(\phi(\lambda),\lambda)]=\int_{\phi(\lambda)-\epsilon(\lambda)}^{\phi(\lambda)+\epsilon(\lambda)}\exp[\tfrac{1}{2}(u-\phi(\lambda))^2F_{uu}(X,\lambda)]du,$$

where $X=X(\lambda)$ and $|X(\lambda)-\phi(\lambda)|<\epsilon(\lambda)$. Set

$$M(\lambda)=\underset{\phi(\lambda)-\epsilon<x<\phi(\lambda)+\epsilon}{\text{l. u. b.}}|F_{uu}(x,\lambda)|,\quad m(\lambda)=\underset{\phi(\lambda)-\epsilon<x<\phi(\lambda)+\epsilon}{\text{g. l. b.}}|F_{uu}(x,\lambda)|.$$

Then by a simple change of variable and upon a comparison with the probability integral we easily obtain

$$(9)\qquad (1-\delta)(2\pi/M(\lambda))^{\frac{1}{2}}\leq J_2(\lambda)\exp[-F(\phi(\lambda),\lambda)]\leq(2\pi/m(\lambda))^{\frac{1}{2}},$$

where

$$(1-\delta)(2\pi)^{\frac{1}{2}}=\int_{-\epsilon(M(\lambda))^{\frac{1}{2}}}^{\epsilon(M(\lambda))^{\frac{1}{2}}}e^{-\frac{1}{2}t^2}dt.$$

Thus $\delta=\delta(\lambda)\to 0$ provided that $\epsilon(\lambda)(M(\lambda))^{\frac{1}{2}}\to\infty$ with $\lambda$. Clearly hypothesis (iii) implies that

$$M(\lambda)/m(\lambda)\to 1,\qquad M(\lambda)/(-F_{uu}(\phi(\lambda),\lambda))\to 1,\qquad(\lambda\to\infty),$$

since $\epsilon(\lambda)\to 0$ and $\phi(\lambda)\to\xi$. Thus in order to prove Lemma 1 we need only show that there exists a positive function $\epsilon(\lambda)$ such that when $\lambda\to\infty$, we have

I) $\epsilon(\lambda)\to 0$, II) $\epsilon(\lambda)(-F_{uu}(\phi(\lambda),\lambda))^{\frac{1}{2}}\to\infty$,

III) $J_1(\lambda)/J_2(\lambda)\to 0$, IV) $J_3(\lambda)/J_2(\lambda)\to 0$.

For the purpose just mentioned we may take

$$(10)\qquad \epsilon(\lambda)=(\log[-F_{uu}(\phi(\lambda),\lambda)]/(-F_{uu}(\phi(\lambda),\lambda)))^{\frac{1}{2}}.$$

That the conditions I) and II) are fulfilled is obvious by our hypothesis (iii). We now only verify the conidtion IV), as the justification of III) is similar.

Obviously for every large $\lambda$, $F(u,\lambda)$ is a monotonic decreasing function in $(\phi(\lambda),\infty)$, for, by hypothesis (ii) we have $F_{uu}(u,\lambda)<0$ and consequently $F_u(u,\lambda)<0$ in $(\phi(\lambda),\infty)$. Thus making change of variable $F(u,\lambda)=-t$ in $J_3$, setting $F(\phi(\lambda)+\epsilon(\lambda),\lambda)=T$ and using the law of the mean we obtain

$$\begin{aligned}0<J_3(\lambda)&\leq\int_{-T}^{+\infty}\{-F_u(\phi(\lambda)+\epsilon(\lambda),\lambda)\}^{-1}\exp(-t)\,dt\\&=\int_{-T}^{+\infty}\{-\epsilon(\lambda)F_{uu}(\phi(\lambda)+\theta\epsilon(\lambda),\lambda)\}^{-1}\exp(-t)\,dt\\&=\{-\epsilon(\lambda)F_{uu}(\phi(\lambda)+\theta\epsilon,\lambda)\}^{-1}\exp F(\phi(\lambda)+\epsilon(\lambda),\lambda)\\&\leq\{m(\lambda)\epsilon(\lambda)\}^{-1}\exp F(\phi(\lambda)+\epsilon(\lambda),\lambda),\end{aligned}$$

where $0<\theta<1$. Thus by (9) and II) we have

$$\begin{aligned}0<\frac{J_3(\lambda)}{J_2(\lambda)}&\leq\frac{\exp[F(\phi(\lambda+\epsilon,\lambda)-F(\phi(\lambda),\lambda)]}{(1-\delta)\cdot m(\lambda)\cdot\epsilon(\lambda)}\left(\frac{M(\lambda)}{2\pi}\right)^{\frac{1}{2}}\\&<A\,\frac{\exp[-\frac{1}{2}\epsilon(\lambda)^2m(\lambda)]}{(m(\lambda))^{\frac{1}{2}}\cdot\epsilon(\lambda)}\to 0,\end{aligned}$$

as $\lambda\to\infty$. In a like manner we may verify III) by setting $F(u,\lambda)=t$. Hence in conclusion the function $\epsilon(\lambda)$ defined by (10) satisfies all the required conditions.

Clearly Lemma 1 is also true for the case $a>-\infty$ or $b<+\infty$, since all the inequalities occurred above still hold.

To illustrate the use of Lemma 1 we give the following simple examples:

1. $$\int_0^\infty u^\lambda\exp(-u\lambda^2/(A\lambda+B\lambda^{\frac{1}{2}}))\,du\sim A(eE)^{-\lambda}(2\pi/\lambda)^{\frac{1}{2}},\qquad(\lambda\to\infty),$$

2. $$\begin{aligned}&\int_0^\infty u^\lambda(1+u)^{-1}\exp(-u\lambda^2/(A\lambda+B\lambda^{\frac{1}{2}}))\,du\\&\sim(1+A)[(\phi(\lambda))^\lambda/(1+\phi(\lambda))]\exp(-\lambda^2\phi(\lambda)/(A\lambda+B\lambda^{\frac{1}{2}}))\\&\qquad\times(2\pi/(\lambda(1+A^{-1})^2-1))^{\frac{1}{2}},\end{aligned}$$

where $A$, $B$ are arbitrary constants with $A>0$, and

$$E=\lambda/(A\lambda+B\lambda^{\frac{1}{2}}),$$
$$\phi(\lambda)=(2E)^{-1}\{[(E-1+1/\lambda)^2+4E]^{\frac{1}{2}}-E+1-1/\lambda\}.$$

The expression 1. can also be verified immediately by use of the Eulerian integral and Stirling's formula. The relation 2. cannot be directly obtained by the classical Laplace asymptotic evaluation.

LEMMA 2. *If $a$, $b$ are finite numbers $(a<b)$ and if in addition to hypotheses* (i), (ii), (iii) *of Lemma* 1 *we have*

(iv) $$g(u)\,\varepsilon\, L(a,b), \qquad g(\xi)\neq 0, \qquad (a<\xi<b),$$

*where $\xi$ belongs to the Lebesgue set for $g(u)$, then there is a sequence of points $\{\phi(\lambda_n)\}$ belonging to the Lebesgue set of $g(u)$ with $\lambda_n\to\infty$, and for $\lambda_n\to\infty$ we have*

$$\int_a^b \exp[F(u,\lambda_n)]g(u)\,du \sim \exp[F(\phi(\lambda_n),\lambda_n)]\,g(\xi)\,(-2\pi/F_{uu}(\xi,\lambda_n))^{1/2}. \tag{11}$$

*Proof.* We have assumed in the hypothesis (i) that $u=\phi(\lambda)$ is a continuous function approaching $\xi(a<\xi<b)$ as $\lambda\to\infty$. Hypothesis (iv) implies that the points of the Lebesgue set for $g(u)$ are almost everywhere in $(a,b)$. Thus by the continuity of $\phi(\lambda)$ it is easy to see that an unbounded increasing sequence $\{\lambda_n\}$ can always be chosen such that $\{\phi(\lambda_n)\}$ belong to the Lebesgue set for $g(u)$.

By use of (8) we need only show that, as $\lambda_n\to\infty$,

$$I=(-F_{uu}(\xi,\lambda_n))^{1/2}\int_a^b[g(u)-g(\xi)]\exp[F(u,\lambda_n)-F(\phi(\lambda_n),\lambda_n)]\,du = o(1).$$

Write

$$I=\int_a^{\phi(\lambda_n)-\delta}+\int_{\phi(\lambda_n)-\delta}^{\phi(\lambda_n)}+\int_{\phi(\lambda_n)}^{\phi(\lambda_n)+\delta}+\int_{\phi(\lambda_n)+\delta}^{b}=J_1+J_2+J_3+J_4,$$

where $\delta>0$ is a properly chosen fixed number. It suffices to prove that $J_3(n)+J_4(n)=o(1)$ as $n\to\infty$; for the other parts, $J_1+J_2$, can be treated in exactly the same manner.

Recall that for every fixed large $v=\lambda_n$, the function $F(u,v)$ has a maximum at $u=\phi(\lambda_n)$, and $F(u,\lambda_n)$ is decreasing in $\phi(\lambda_n)\leq u\leq b$. Hence

$$\begin{aligned}|J_4(n)| &\leq (-F_{uu}(\xi,\lambda_n))^{1/2}\exp[F(\phi(\lambda_n)+\delta,\lambda_n)\\ &\quad -F(\phi(\lambda_n),\lambda_n)]\int_{\phi(\lambda_n)+\delta}^{b}|g(u)-g(\xi)|\,du\\ &\leq (-F_{uu}(\xi,\lambda_n))^{1/2}\exp[\tfrac{1}{2}\delta^2F_{uu}(\phi+\theta\delta,\lambda_n)]A,\end{aligned}$$

where $0<\theta<1$, and $A$ is implied by the fact that $g\,\varepsilon\,L(a,b)$.

Clearly hypothesis (iii) implies the existence of a small number $\delta_1>0$ and a large number $N_1$ such that, e. g.

$$0<-\tfrac{3}{4}F_{uu}(\xi,\lambda_n)<-F_{uu}(u,\lambda_n)<-\tfrac{5}{4}F_{uu}(\xi,\lambda_n), \tag{12}$$

whenever $|u-\xi|<\delta_1$, and $\lambda_n>N_1$. Since $\phi(\lambda_n)\to\xi$ we may choose $M$ so large and $\delta$ so small that $|\phi(\lambda_n)+\theta\delta-\xi|<\delta_1$, whenever $\lambda_n>M$,

where $\delta$ is fixed after choice. Thus it follows from (iii) and (12) that $-F_{uu}(\phi+\theta\delta, \lambda_n) > -\frac{3}{4}F_{uu}(\xi, \lambda_n) \to \infty$ as $\lambda_n \to \infty$, and consequently,

$$\overline{\lim_{n\to\infty}} |J_4(n)| \leq \lim_{n\to\infty} A(-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}} \exp[\tfrac{3}{8}\delta^2 F_{uu}(\xi, \lambda_n)] = 0$$

Hence in conclusion $J_4(n) = o(1)$.

Now consider $J_3(n)$. Since $\xi, \phi(\lambda_n)$ are points belonging to the Lebesgue set for $g(u)$, we have

$$\int_\xi^u |g(x) - g(\xi)|\, dx = o(|u-\xi|), \qquad (|u-\xi| \to 0);$$

(13)

$$\int_\phi^w |g(x) - g(\phi)|\, dx = o(|w-\phi|), \qquad (|w-\phi| \to 0).$$

Putting $u = \phi(\lambda_n)$, $w = \xi$ in (13) and comparing, we get

$$\int_\xi^\phi |g(\phi) - g(\xi)|\, dx = o(|\phi(\lambda_n) - \xi|), \qquad (|\phi - \xi| \to 0).$$

This is equivalent to $|g(\phi) - g(\xi)| = o(1)$. That means $g(\phi(\lambda_n)) \to g(\xi)$ as $\lambda_n \to \infty$. Set

$$\alpha(u) = \int_{\phi(\lambda_n)}^u [g(x) - g(\phi(\lambda_n))]dx = o(|u - \phi(\lambda_n)|), \quad (u \to \phi(\lambda_n)),$$

and write $g(u) - g(\xi) = [g(w) - g(\phi)] + K(\phi(\lambda_n))$; so that $\alpha(\phi(\lambda_n)) \equiv 0$ and $K(\phi) = g(\phi(\lambda_n)) - g(\xi) \to 0$ as $\lambda_n \to \infty$. Then integration by parts gives

$$\begin{aligned} J_3(n) &= (-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}} \int_\phi^{\phi+\delta} \{K(\phi) + [g(u) - g(\phi)]\} \exp\{F(u, \lambda_n) - F(\phi, \lambda_n)\} du \\ &= (-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}} \int_\phi^{\phi+\delta} K(\phi) \exp\{F(u, \lambda_n) - F(\phi, \lambda_n)\} du \\ &\quad + (-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}} \alpha(\phi+\delta) \exp\{F(\phi+\delta, \lambda_n) - F(\phi, \lambda_n)\} \\ &\quad - (-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}} \int_\phi^{\phi+\delta} \alpha(u) \exp\{F(u, \lambda_n) - F(\phi, \lambda_n)\} F_u(u, \lambda_n) du \\ &= J_5(n) + J_6(n) + J_7(n). \end{aligned}$$

Here $J_6(n) = o(1)$ $(n \to \infty)$ is easily inferred by the same argument as used in proving $J_4(n) = o(1)$. For proof of $J_5(n) = o(1)$ and $J_7(n) = o(1)$, we write

$$\begin{aligned} F(u, \lambda_n) - F(\phi, \lambda_n) &= \tfrac{1}{2}(u-\phi)^2 F_{uu}(\xi', \lambda_n), \\ F_u(u, \lambda_n) &= (u-\phi) F_{uu}(\xi'', \lambda_n), \end{aligned} \qquad (14)$$

where $\xi' = \phi(\lambda_n) + \theta'(u - \phi(\lambda_n))$, $\xi'' = \phi(\lambda_n) + \theta''(u - \phi(\lambda_n))$, $0 < \theta'$,

$\theta'' < 1$, $\phi \leq u \leq \phi + \delta$. Clearly $\xi' \cong \xi + \theta'(u - \xi)$, $\xi'' \cong \xi + \theta''(u - \xi)$ as $\lambda_n \to \infty$, so that by (iii) or by the same reasoning of (12) we have, for $\lambda_n$ sufficiently large and for certain fixed small $\delta > 0$,

$$(15) \qquad -F_{uu}(\xi', \lambda_n) > -A_1 F_{uu}(\xi, \lambda_n),$$
$$-A_2 F_{uu}(\phi, \lambda_n) > -F_{uu}(\xi'', \lambda_n) > -A_3 F_{uu}(\phi, \lambda_n),$$

where $A_1$, $A_2$, $A_3$ are certain positive numbers. Now for any given $\epsilon > 0$ we can determine a large number $M'$ and a small positive number $\eta$ ($\leq \delta$) such that

$$(16) \qquad |\alpha(u)| < \epsilon |u - \phi|, \qquad |K(\phi)| < \epsilon,$$

for $|u - \phi(\lambda_n)| < \eta$ and $\lambda_n > M'$. It is obvious that $J_4(n) = o(1)$, $J_6(n) = o(1)$ are also true when $\delta$ is replaced by $\eta$. Thus using (14), (15), (16) and replacing $\delta$ by $\eta$, we get

$$|J_5(n)| < \epsilon \int_\phi^{\phi+\eta} (-F_{uu}((\xi, \lambda_n))^{\frac{1}{2}} \exp\{F(u, \lambda_n) - F(\phi, \lambda_n)\} du$$
$$< \epsilon \int_\phi^{\phi+\eta} (-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}} \exp\{\tfrac{1}{2} A_1 (u - \phi)^2 F_{uu}(\xi, \lambda_n)\} du$$
$$\leq \epsilon \int_0^\infty \exp\{-\tfrac{1}{2} A_1 t^2\} dt = A\epsilon,$$

with $t = (-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}}(u - \phi)$, and similarly,

$$|J_7(n)| < \epsilon(-F_{uu}(\xi, \lambda_n))^{\frac{1}{2}} \int_\phi^{\phi+\eta} (u - \phi)^2 |F_{uu}(\xi'', \lambda_n)| \exp\{\tfrac{1}{2}(u - \phi)^2 F_{uu}(\xi'', \lambda_n)\} du$$
$$< A\epsilon \int_\phi^{\phi+\eta} (-F_{uu}(\phi, \lambda_n))^{3/2} (u - \phi(\lambda_n))^2 \exp\{\tfrac{1}{2} A_3 (u - \phi)^2 F_{uu}(\phi, \lambda_n)\} du$$
$$\leq A\epsilon \int_0^\infty s^2 \exp\{-\tfrac{1}{2} A_3 s^2\} ds = A\epsilon,$$

with $s = (-F_{uu}(\phi, \lambda_n))^{\frac{1}{2}}(u - \phi(\lambda_n))$. Hence it follows that $\overline{\lim} |J_5(n)| \leq A\epsilon$, $\overline{\lim} |J_7(n)| \leq A\epsilon$ where $n \to \infty$. Since $\epsilon$ is arbitrary we get $J_5(n) = o(1)$, $J_7(n) = o(1)$ $(n \to \infty)$. This completes the proof of $J_3(n) + J_4(n) = o(1)$. By a similar procedure it can be shown that $J_1(n) + J_2(n) = o(1)$. Therefore our Lemma 2 is true.

LEMMA 3. *Let $F(u, v)$ satisfy all hypotheses of Lemma 1 with $a = 0$, $b = \infty$, $0 < \xi < \infty$, and let $g(u) \in L(0, \infty)$ with $g(\xi) \neq 0$, $\xi$ being a point of the Lebesgue set for $g(u)$. Then we have*

$$(17) \qquad \int_0^\infty \exp[F(u, \lambda_n)] g(u) du$$
$$\sim \exp[F(\phi(\lambda_n), \lambda_n)] g(\xi) (-2\pi / F_{uu}(\xi, \lambda_n))^{\frac{1}{2}},$$

*where $\{\lambda_n\}$ is an unbounded increasing sequence for which $\{\phi(\lambda_n)\}$ belong to the Lebesgue set for $g(u)$. Moreover if $g(\xi) = 0$, then*

$$\int_0^\infty \exp[F(u, \lambda_n)]g(u)\,du = o(\exp[F(\phi(\lambda_n), \lambda_n)](-2\pi/F_{uu}(\xi, \lambda_n))^{\frac{1}{2}}. \tag{18}$$

*Proof.* By use of Lemma 2 we need only to show that for any fixed number $\delta > 0$, we have, for $n \to \infty$,

$$I(n) = (F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}} \int_{\xi+\delta}^{x} |g(u)| \exp\{F(u, \lambda_n) - F(\phi(\lambda_n), \lambda_n)\}du = o(1).$$

Set $\alpha(x) = \int_{\xi+\delta}^{x} |g(u)|\,du$, so that $\alpha(\xi + \delta) = 0$, and $\alpha(\infty)$ is a finite value. Integration by parts gives

$$\begin{aligned} I(n) &= (F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}}\alpha(\infty) \exp\{F(\infty, \lambda_n) - F(\phi, \lambda_n)\} \\ &\quad - (F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}} \int_{\xi+\delta}^{\infty} \exp\{F(u, \lambda_n) - F(\phi, \lambda_n)\}\alpha(u)F_u(u, \lambda_n)du \\ &= J_1(n) - J_2(n), \end{aligned}$$

where

$$F(\infty, \lambda_n) - F(\phi, \lambda_n) = \int_{\phi(\lambda_n)}^{\infty} F_u(u, \lambda_n)\,du. \tag{19}$$

Obviously $F_u(u, \lambda_n) \leq 0$ is a monotonic decreasing function of $u$ in $(\phi, \infty)$ for sufficiently large $\lambda_n$, so that in case the integral of (19) does not converge we must have $F(\infty, \lambda_n) = -\infty$. We shall now prove that $J_1(n) = o(1)$, $J_2(n) = o(1)$ $(n \to \infty)$ without assuming $F(\infty, \lambda_n) = -\infty$. Since $\alpha(u)$ is bounded we have

$$\begin{aligned} |J_2(n)| &\leq A(F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}} \left| \int_{\xi+\delta}^{\infty} \exp\{F(u, \lambda_n) - F(\phi, \lambda_n)\}F_u(u, \lambda_n)du \right| \\ &= A(F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}} |\exp\{F(\infty, \lambda_n) \\ &\qquad - F(\phi, \lambda_n)\} - \exp\{F(\xi + \delta, \lambda_n) - F(\phi, \lambda_n)\}|. \end{aligned}$$

Therefore in order to establish $J_1(n) = o(1)$ and $J_2(n) = o(1)$ we need only to show that, as $n \to \infty$,

$$\lim (F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}} \exp\{F(\xi + \delta, \lambda_n) - F(\phi, \lambda_n)\} \to 0, \tag{20}$$

$$\lim (F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}} \exp\{F(\infty, \lambda_n) - F(\phi, \lambda_n)\} \to 0. \tag{21}$$

Note that $F(\xi + \delta, \lambda_n) - F(\phi, \lambda_n) = \frac{1}{2}\Delta^2 F_{uu}(\phi + \theta\Delta, \lambda_n)$, where $o < \theta < 1$ and $\Delta = \xi + \delta - \phi(\lambda_n) \to \delta > 0$. Thus by choosing suitable fixed $\delta > 0$ and using the same argument as previously employed in proving $J_4(n) = o(1)$

(see Proof of Lemma 2), we have, when $\lambda_n \to \infty$, $-F_{uu}(\phi + \theta\Delta, \lambda_n) > -\frac{3}{4}F_{uu}(\xi, \lambda_n) \to \infty$, and consequently,

$$(F_{uu}(\xi, \lambda_n)/(-2\pi))^{\frac{1}{2}}\exp\{\tfrac{1}{2}\Delta^2 F_{uu}(\phi + \theta\Delta, \lambda_n)\} = o(1).$$

This establishes (20).

To prove (16), let us notice that

$$F_u(u, \lambda_n) = \int_{\phi(\lambda_n)}^{u} F_{uu}(t, \lambda_n)\,dt < 0,$$

whenever $u > \phi(\lambda_n)$, using hypothesis (ii). Thus for $n$ so large that that $\xi + \delta \geq \phi(\lambda_n)$ we have

$$F(\infty, \lambda_n) = F(\xi + \delta, \lambda_n) + \int_{\xi+\delta}^{\infty} F_u(u, \lambda_n)\,du < F(\xi + \delta, \lambda_n).$$

This shows clearly that (21) is implied by (20). Hence in conclusion $I(n) = o(1)$, and the formula (17) follows from Lemma 2. Moreover if we take $g(u) = g_1(u) + g_2(u)$ with $g_1(\xi) \not\equiv 0$, $g_2(\xi) = 0$, then an application of (17) yields the equation (18) with $g(u) = g_2(u)$. Our lemma is thus completely proved.

*Example.* Evidently $(1 + u^\alpha)^{-1} \in L(0, \infty)$ with $\alpha > 1$. Thus by Example 1. of Lemma 1 and by Lemma 3 we have, as $\lambda \to \infty$,

3. $$\int_0^\infty u^\lambda/(1 + u^\alpha)\exp(-u\lambda^2/(A\lambda + B\lambda^{\frac{1}{2}}))\,du \sim (A/(1+A^\alpha))(A + B/\lambda^{\frac{1}{2}})^\lambda e^{-\lambda}(2\pi/\lambda)^{\frac{1}{2}}.$$

It is noteworthy that the above asymptotic relation cannot be directly obtained from (8) of Lemma 1, as the integrand function does not satisfy conditions (i), (ii), (iii). If $\alpha = 1$, then Lemma 1 is applicable and we have Example 2.

Clearly Lemmas 2 and 3 are extensions of the classical Laplace asymptotic evaluation for an integral ([4; p. 78], [1; pp. 277-278]).

**3. A general inversion theorem.** Using the results obtained in § 2 we may now state sufficient conditions for the validity of the general inversion formula (5). For convenience we introduce the following definitions:

DEFINITION 1. *A function $\psi(u, v, t)$ is called a regular transform function with respect to a certain fixed $t = t_1 > 0$, if $\log \psi(u, v, t_1) = F(u, v, t_1)$, $F_u(u, v, t_1)$, $F_{uu}(u, v, t_1)$ are real continuous functions $(u \geq 0,\ v > 0)$ satisfying the following conditions:*

i) $F_u(u, v, t_1) = 0$ *has a continuous solution* $u = \phi(v, t_1)$ *with* $\lim_{v\to\infty} \phi(v, t_1) = \theta(t_1) > 0$,

ii) $F_{uu}(u, v, t_1) < 0$ *for all* $u > 0$ *and large* $v$,

iii) $F_{uu}(u, v, t_1) \sim F_{uu}(\theta(t_1), v, t_1) \to -\infty$ *as* $u \to \theta(t_1)$, $v \to \infty$.

A simple example of a regular transform function is

$$\psi(u, v, t) = u^v \exp\{-uv^2/(vt + v^{\frac{1}{2}} \sin t)\}.$$

Of course we may construct as many such transform functions as we like.

DEFINITION 2. $\lim \phi(v, t) = \theta(t)$ *is called a limiting solution of the equation* $(\partial/\partial u)\log \psi(u, v, t) = 0$.

Clearly for every fixed $t > 0$, the functions $F(u, v, t)$, $F_{uu}(u, v, t)$, $F_{uu}(u, v, t)$ etc. can be considered as functions of the two variables $u$, $v$. Thus by Lemmas 1 and 3 we easily establish the following theorem.

THEOREM 1. *If* $\psi(u, v, t) = \exp F(u, v, t)$ *is a regular transform function with respect to a certain* $t > 0$, *if* $g(u) \in L(0, \infty)$, *and if* $f(\lambda, t)$ *is defined by* (4), *then*

$$g(\theta(t)) = \lim_{\lambda\to\infty} \left(\frac{-F_{uu}(u, \lambda, t)}{2\pi(\psi(u, \lambda, t))^2}\right)^{\frac{1}{2}}_{u=\phi(\lambda,t)} f(\lambda, t), \tag{5 bis}$$

*where* $\lambda$ *tends to* $\infty$ *through such a sequence* $\{\lambda_n\}$ *that* $\{\phi(\lambda_n, t)\}$ *together with its limit* $\lim \phi = \theta(t)$ *belong to the Lebesgue set for* $g(u)$.

*Proof.* Clearly in the formulas (17) and (18) we may replace $F_{uu}(\xi, \lambda_n)$ by $F_{uu}(\phi(\lambda_n), \lambda_n)$. Let $F(u, v) = F(u, v, t)$ so that $\exp F(u, v) = \psi(u, v, t)$. where $t$ is fixed. Then from (17) we may infer that, as $\lambda_n \to \infty$,

$$\frac{(-F_{uu}(\phi(\lambda_n, t), \lambda_n, t))^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} \exp F(\phi(\lambda_n, t), \lambda_n, t)} \int_0^\infty \psi(u, \lambda_n, t) g(u)\, du \to g(\theta(t)),$$

where $\phi(\lambda_n, t) = \phi(\lambda_n)$, $\theta(t) = \xi$, $g(\xi) \neq 0$. This is just equivalent to (5 bis) with $g(\theta(t)) \neq 0$. Moreover if $g(\theta(t)) = 0$, then by (18) we have, for $\lambda_n \to \infty$,

$$\frac{(-F_{uu}(\phi(\lambda_n, t), \lambda_n, t))^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} \exp F(\phi(\lambda_n, t), \lambda_n, t)} \int_0^\infty \psi(u, \lambda_n, t) g(u)\, du = o(1).$$

Hence in either case the formula (5 bis) is true.

COROLLARY. *If* $\psi(u, v, t)$ *is a regular transform function and if* $g(u)$

*is a continuous function, absolutely integrable over* $(0, \infty)$, *then* (5 bis) *is valid, where* $\lambda$ *tends to* $\infty$ *through any sequence.*

**4. Specialization.** We now consider transform functions of the form

$$\psi(u, \lambda, t) = \exp\{\lambda \Phi(u, t)\}, \qquad (u \geq 0, t > 0), \tag{22}$$

where $\Phi(u, t)$ is a real valued function. If $\Phi_u(u, t)$, $\Phi_{uu}(u, t)$ are continuous functions such that

i)[1] there is a continuous function $u = \xi(t) > 0$ satisfying the equation $\Phi_u(\xi(t), t) = 0$.

ii)[1] $\Phi_{uu}(u, t) < 0$ for all $u, t > 0$.

then clearly $\psi(u, \lambda, t)$ is a regular transform function, because the condition iii) of Definition 1 of § **3** is automatically satisfied. Obviously in the present case our formula (5 bis) takes the form

$$g(\xi(t)) = \lim_{\lambda \to \infty} \frac{[-\lambda \Phi_{uu}(\xi(t), t)]^{\frac{1}{2}}}{(2\pi)^{\frac{1}{2}} \exp[\lambda \Phi(\xi(t), t)]} f(\lambda, t), \tag{5}[1]$$

where $f(\lambda, t)$ is defined by (4), and $\lambda \to \infty$ through any real sequence.

As another particular application of Theorem 1, we take

$$\psi(u, \lambda, t) = \exp\{-\lambda u \beta(\lambda, t)\} u^{\lambda}, \tag{23}$$

where $\beta(\lambda, t)$ is an arbitrary continuous function with $\beta(\lambda, t) > 0$, $\lim \beta(\lambda, t) > 0$ $(\lambda \to \infty)$. Obviously we have

$$\int_0^{\infty} g(u) \exp\{-\lambda u \beta(\lambda, t)\} u^{\lambda} \, du = (-1)^{\lambda} f^{(\lambda)}(\lambda \beta(\lambda, t)),$$

where $f(x)$ is the Laplace transform of $g(u)$ (cf. (3)). Thus using (5) or (5 bis) we easily obtain

$$g(\alpha(t)) = \lim_{\lambda \to \infty} (-1)^{\lambda} f^{(\lambda)}(\lambda \beta(\lambda, t)) (\lambda \beta(\lambda, t))^{\lambda+1} / \lambda!, \tag{24}$$

where $g(u) \in L(0, \infty)$, $\alpha(t) > 0$, $\lim \alpha(t) \beta(\lambda, t) = 1$ $(\lambda \to \infty)$ and $\lambda \to \infty$ through any sequence such that $\{\beta(\lambda, t)^{-1}\}$ belongs to the Lebesgue set for $g(u)$.

Recall that the continuity condition imposed on $\phi(\lambda, t) = \beta(\lambda, t)^{-1}$ is only used for asserting the existence of $\{\phi(\lambda_n, t)\}_{n=1}^{\infty}$ (see Proof of Lemma 2), so that it may be replaced by a direct assumption on $\{\beta(\lambda_n, t)^{-1}\}_{n=1}^{\infty}$. Thus by taking $\alpha(t) = t$, $\beta(\lambda, t) = 1/\phi(t)$ in the equation (24), we have

THEOREM 2. *Let $f(x)$ be the Laplace transform of $g(u)$ with $g(u) \in L(0, \infty)$, and let a sequence of points $\{\phi(n)\}$ together with its limit $\lim \phi(n) = t > 0$ belong to the Lebesgue set for $g(u)$. Then*

$$g(t) = \lim_{n\to\infty} (-1)^n f^{(n)}(n/\phi(n))(n/\phi(n))^{n+1}/n!. \tag{25}$$

This theorem may be regarded as an analogue of the result in Widder's book [1] (cf. Theorem 6b of [1; pp. 289-290]).

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# ON THE SPHERICAL CONVERGENCE OF MULTIPLE FOURIER SERIES.*

By JOSEPHINE MITCHELL.

**1. Introduction.** In this paper we propose to use the methods and results of the lattice point problem for the hypersphere in number theory to obtain new results for the spherical convergence of multiple Fourier series. The set of functions $\{(2\pi)^{-q}e(n_1x_1+\cdots+n_qx_q)$ $(n_k=0,\pm 1,\pm 2,\cdots;$ $k=1,\cdots,q)$ (where $e(X)=e^{iX}$ throughout this paper), forms an orthonormal system on the hypercube $C(-\pi\leq x_k\leq\pi;\ k=1,\cdots,q)$, that is,

$$(1.1)\quad (2\pi)^{-q}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}e(m_1x_1+\cdots+m_qx_q)e(-n_1x_1-\cdots-n_qx_q)\times dx_1\cdots dx_q = \prod_{k=1}^{q}\delta_{m_kn_k},$$

which is complete with respect to real Lebesgue integrable functions defined on $C$. We form the series

$$(1.2)\quad \sum a_{n_1\ldots n_q}e(n_1x_1+\cdots+n_qx_q)\ (n_k=-\infty \text{ to } \infty;\ k=1,\cdots,q),$$

where $\{a_{n_1\ldots n_q}\}$ is a sequence of complex numbers. Summing (1.2) by spherical, instead of rectangular, partial sums, we consider the limit of the sequence

$$(1.3)\qquad S_R=\sum_{\nu\leq R}a_{n_1\ldots n_q}e(n_1x_1+\cdots+n_qx_q)\quad(\nu=n_1^2+\cdots+n_q^2)$$

as $R\to\infty$, $R$ being the sequence of integers, which can be represented as the sum of $q$ squares of integers, so that for $q\geq 4$, $R$ takes on all non-negative integral values [6].[1] We ask ourselves what condition must be imposed on the sequence $\{a_{n_1\ldots n_q}\}$ in order that $\lim_{R\to\infty}S_R$ exist and show that *if*[2]

$$\sum(n_1^2+\cdots+n_q^2)^p\,|\,a_{n_1\ldots n_q}\,|^2<\infty,$$

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* Received June 5, 1950; presented to the American Mathematical Society, April 28, 1950.

[1] The numbers in brackets refer to the references at the end of the paper.

[2] The summation for this series is always $n_k=-\infty$ to $+\infty$, $k=1,\cdots,q$, and will be omitted.

$\lim_{R\to\infty} S_R$ *exists almost everywhere on* $C$, *where* $p = 1/3$ *for* $q = 2$ *and* $p = \frac{1}{4}q$ *for* $q \geq 4$, *while for* $q = 3$ *the sequence* $(n_1^2 + n_2^2 + n_3^2)^p$ *is replaced by* $(n_1^2 + n_2^2 + n_3^2)^{2/3}(\log(n_1^2 + n_2^2 + n_3^2)^{5/3}$. This theorem represents a considerable improvement over known results due to Chandrasekharan and Minakshisundaran, where $p = \frac{1}{2}q$ [1, 2]. We prove it by finding a bound for the Lebesgue function

$$(1.4)\quad L_R(\alpha) = L_R(\alpha_1, \cdots, \alpha_q) = \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |K_R(\alpha - x)|\, dx_1 \cdots dx_q,$$

where

$$(1.5)\quad K_R(\alpha - x) = K_R(\alpha_1 - x_1, \cdots, \alpha_q - x_q) = 1/(2\pi)^q \sum_{\nu \leq R} e(n_1(\alpha_1 - x_1) + \cdots + n_q(\alpha_q - x_q)),$$

and then proceeding by classical methods used in Fourier and orthogonal series to the conclusion concerning $\lim S_R$. Finer results are obtained for $q = 2$ and 3, namely, $L_R(\alpha) = O(R^{1/3})$ and $O(R^{2/3}(\log R)^{5/3})$ respectively, by using new results on exponential sums due to L. K. Hua [3] in the lattice point problem for the circle and sphere.

## 2. The case $q = 2$.

### 2.1. We prove

THEOREM 2.1. *The Lebesgue function* $L_R(\alpha, \beta)$ *is* $O(R^{1/3})$.

*Proof.* From the periodicity of the exponential system

$$L_R(\alpha, \beta) = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |K_R(x, y)|\, dx\, dy.$$

In the proof of this theorem we proceed as far as possible by means of the method used in the rectangular summation of Fourier series for the corresponding problem, but in order to complete the analysis some number-theoretic results are introduced (cf. Section **2.3**). Thus we subdivide the square $(-\pi \leq x \leq \pi,\ -\pi \leq y \leq \pi)$ into smaller rectangles, on each of which a suitable bound for $K_R(x, y)$ is utilized, by drawing lines parallel to the axes through the points $x$ and $y = -\pi + \epsilon, -\epsilon, 0, \epsilon, \pi - \epsilon$, where $\epsilon$ is an arbitrary small positive number. The square is divided into rectangles of 3 different types, of which $(0 \leq x \leq \epsilon, 0 \leq y \leq \epsilon)$, $(0 \leq x \leq \epsilon, \epsilon \leq y \leq \pi - \epsilon)$ and $(\epsilon \leq x \leq \pi - \epsilon, \epsilon \leq y \leq \pi - \epsilon)$ are examples. (The fact that $|x - y|$

$< 2\pi$ is used in the discussion of (2.17).) Over squares of side $\epsilon$ we use the bound $|K_R(x,y)| \leq \sum_{m^2+n^2\leq R} 1 = O(R)$ [6], to obtain, for example,

$$\int_0^\epsilon \int_0^\epsilon |K_R(x,y)|\,dx\,dy = O(R\epsilon^2). \tag{2.1}$$

For rectangles of dimension $\epsilon$ times $\pi - 2\epsilon$, if $|y| \geq \epsilon$, we proceed as follows, $[z]$ being the integral part of $z$,

$$\begin{aligned} K_R(x,y) &= \sum_{-R^{1/2}\leq m\leq R^{1/2}} e(mx) \sum_{n=-[(R-m^2)^{1/2}]}^{[(R-m^2)^{1/2}]} e(ny) \\ &= 1/(e(y)-1) \sum_{-R^{1/2}\leq m\leq R^{1/2}} e(mx)\{e([(R-m^2)^{\frac{1}{2}}]+1)y) \\ &\qquad - e(-[(R-m^2)^{\frac{1}{2}}]y)\} \\ &= O(R^{\frac{1}{2}} \csc \tfrac{1}{2}y). \end{aligned} \tag{2.2}$$

Hence, for example,

$$\int_\epsilon^{\pi-\epsilon} dy \int_0^\epsilon |K_R(x,y)|\,dx = O(\epsilon R^{\frac{1}{2}} \log \epsilon^{-1}). \tag{2.3}$$

Similarly

$$\begin{aligned} &\int_\epsilon^{\pi-\epsilon} dy \int_{y-\frac{1}{2}\epsilon}^{y+\frac{1}{2}\epsilon} |K_R(x,y)|\,dx \\ &\quad = O(R^{\frac{1}{2}} \int_\epsilon^{\pi-\epsilon} \csc \tfrac{1}{2}y\,dy \int_{y-\frac{1}{2}\epsilon}^{y+\frac{1}{2}\epsilon} dx) = O(R^{\frac{1}{2}}\epsilon \log \epsilon^{-1}). \end{aligned} \tag{2.4}$$

(Also, of course, $\int_\epsilon^{\pi-\epsilon} dx \int_{x-\frac{1}{2}\epsilon}^{x+\frac{1}{2}\epsilon} |K_R(x,y)|\,dy = O(R^{\frac{1}{2}}\epsilon \log \epsilon^{-1}))$; a result which we shall need later. In the course of our argument we note that all bounds introduced except those involving $O(|x-y|^{-1})$ may be integrated over $(\epsilon \leq y \leq \pi - \epsilon,\ y - \frac{1}{2}\epsilon \leq x \leq y + \frac{1}{2}\epsilon)$ as well as over $(\epsilon \leqq x \leqq \pi - \epsilon,\ \epsilon \leq y \leq \pi - \epsilon)$, obtaining the same or even sharper results.

**2.2.** In order to discuss the sum $\sum_{m^2+n^2\leq R}$ over rectangles of the third type we subdivide the circle $m^2 + n^2 \leq R$ by the lines $n = m$, $n = -m$, $m = 0$ and $n = 0$. On the line $n = m$, for example, we have

$$\begin{aligned} K_R(x,y) &= \sum_{m=-[(\frac{1}{2}R)^{1/2}]}^{[(\frac{1}{2}R)^{1/2}]} e((x+y)m) \\ &= \{e((x+y)([(\tfrac{1}{2}R)^{\frac{1}{2}}]+1)) \\ &\qquad - e(-(x+y)[(\tfrac{1}{2}R)^{\frac{1}{2}}])\}/\{e(x+y)-1\} \end{aligned} \tag{2.5}$$

$= Q_1$. Hence

$$(2.6)\qquad \int_\epsilon^{\pi-\epsilon}\int_\epsilon^{\pi-\epsilon} |\,Q_1\,|\,dx\,dy = O\Big(\int_\epsilon^{\pi-\epsilon}\int_\epsilon^{\pi-\epsilon}(x+y)^{-1}dx\,dy\Big)$$
$$= O\Big(\int_\epsilon^{\pi-\epsilon}\log(x+y)\Big[_{y=\epsilon}^{\pi-\epsilon}dx\Big),$$

and the second mean value theorem used on the latter term gives

$$O(\log(\epsilon+y)\int_\epsilon^{x_0}dx + \log(\pi-\epsilon+y)\int_{x_0}^{\pi-\epsilon}dx\Big[_{y=\epsilon}^{\pi-\epsilon}\Big)\ (\epsilon\le x_0\le\pi-\epsilon),$$

which is $O(|\log\epsilon|)$. On the other hand, for example, for the rectangle $(\epsilon\le x\le\pi-\epsilon, -\pi+\epsilon\le y\le-\epsilon)$, we use the procedure in (2.4) over $(\epsilon\le x\le\pi-\epsilon, -x-\frac{1}{2}\epsilon\le y\le -x+\frac{1}{2}\epsilon)$ and then, as is (2.6)

$$(2.7)\qquad \Big\{\int_\epsilon^{\pi-1.5\epsilon}\int_{-\pi+\epsilon}^{-x-\frac{1}{2}\epsilon} + \int_{1.5\epsilon}^{\pi-\epsilon}\int_{-\pi+\epsilon/2}^{-\epsilon}\Big\}\,|\,Q_1\,|\,dx\,dy = O(\log\epsilon^{-1}).$$

Analogous results are obtained when $m=-n$, $m=0$ or $n=0$.

The remaining sums to be considered in $\sum_{m^2+n^2\le R}$ are all of the form

$$(2.8)\qquad \sum_{0<m\le a}\ \sum_{m<n\le(R-m^2)^{1/2}} e(\pm mx\pm ny)$$
$$(\epsilon\le x\le\pi-\epsilon, \epsilon\le y\le\pi-\epsilon)\ (a=(\tfrac{1}{2}R)^{\frac{1}{2}}),$$

or this sum with $m$ and $n$ interchanged, of which, as we shall see, it is sufficient to study the cases $mx+ny$ and $-mx+ny$. Hence consider first

$$(2.9)\qquad S = \sum_{0<m\le a}\ \sum_{m<n\le(R-m^2)^{1/2}} e(mx+ny)$$
$$= \frac{e(y)}{e(y)-1}\sum_{0<m\le a} e([(R-m^2)^{\frac{1}{2}}]y+mx)$$
$$-\frac{e(y)}{e(y)-1}\sum_{0<m\le a} e((x+y)m)$$

$= S_1+S_2$. Now

$$S_2 = (e((x+y)[a])-1)(e(y)-1)^{-1}(e(x+y)-1)^{-1}e(x+2y)$$

$= O(\csc\frac{1}{2}y\csc\frac{1}{2}(x+y))$, so that

$$(2.10)\qquad \int_\epsilon^{\pi-\epsilon}\int_\epsilon^{\pi-\epsilon}|\,S_2\,|\,dx\,dy = O\Big(\iint_{2\epsilon\le x+y\le\pi} y^{-1}(x+y)^{-1}dx\,dy$$
$$+\iint_{\pi\le x+y\le 2\pi-2\epsilon} y^{-1}(2\pi-x-y)^{-1}dx\,dy\Big)$$
$$= O\Big(\int_\epsilon^{\pi-\epsilon}\int_\epsilon^{\pi-\epsilon}(x+y)^{-1}\{y^{-1}+(\pi-y)^{-1}\}dx\,dy\Big)$$
$$= O\Big(\int_\epsilon^{\pi-\epsilon}\{y^{-1}+(\pi-1)^{-1}\}\log(x+y)\,dy\Big[_{x=\epsilon}^{\pi-\epsilon}\Big)$$

and using the second mean value theorem on $\log(x+y)$, we get (2.10) to be $O(\log^2 \epsilon^{-1})$.

Since the exponent $[(R-m^2)^{\frac{1}{2}}]$ in $S_1$ is discontinuous, this sum cannot be dealt with by standard methods. Consequently we replace $[(R-m^2)^{\frac{1}{2}}]$ by $(R-m^2)^{\frac{1}{2}}-\frac{1}{2}$ and consider

$$S_{11}=\frac{e(y)}{e(y)-1}\sum_{0<m\leq a} e(mx+y(R-m^2)^{\frac{1}{2}}-\tfrac{1}{2}y) \tag{2.11}$$

and

$$S_{12}=\frac{e(y)}{e(y)-1}\sum_{0<m\leq a}\{e(mx+y[(R-m^2)^{\frac{1}{2}}]) - e(mx+y(R-m^2)^{\frac{1}{2}}-\tfrac{1}{2}y)\}, \tag{2.12}$$

where $S_{11}+S_{12}=S_1$. The exponent $2\pi f(u)=ux+y(R-u^2)^{\frac{1}{2}}$ in $S_{11}$ satisfies the hypotheses of the following theorem which is due to van der Corput [8]:

*Let* $D=\int_0^a e(2\pi f(u))\,du-\sum_{0<m\leq a} e(2\pi f(m))$. *If* $f'(u)$ *is monotonic and* $|f'(u)|\leq\frac{1}{2}$, *then* $D=O(1)$. Hence

$$S_{11}(e(y)-1)e(-\tfrac{1}{2}y)=\int_0^a e(ux+y(R-u^2)^{\frac{1}{2}})\,du+O(1) = I_1+O(1). \tag{2.13}$$

Setting $rv=xu+y(R-u^2)^{\frac{1}{2}}$ in $I_1$, where $r=(x^2+y^2)^{\frac{1}{2}}$ and $ru=xv-y(R-v^2)^{\frac{1}{2}}$, we get

$$I_1=r^{-1}\int_{v_1}^{v_2}(x+yv/(R-v^2)^{\frac{1}{2}})e(rv)\,dv=I_{11}+I_{12} \tag{2.14}$$

($v_1=yR^{\frac{1}{2}}r^{-1}$, $v_2=(x+y)ar^{-1}$). Now $I_{11}=O(x(x^2+y^2)^{-1})$ so that as in (2.10)

$$\int_\epsilon^\pi\int_\epsilon^\pi \csc\tfrac{1}{2}y\,|\,I_{11}\,|\,dx\,dy=O(\log^2\epsilon^{-1}). \tag{2.15}$$

The function $g(v)=v(R-v^2)^{-\frac{1}{2}}$ in the integral $I_{12}$ is montone and $g(v_1)=yx^{-1}$, $g(v_2)=(x+y)\,|\,x-y\,|^{-1}$ so that by the second mean value theorem $I_{12}=O(y^2x^{-1}(x^2+y^2)^{-1})+O(y(x+y)\,|\,x-y\,|^{-1}(x^2+y^2)^{-1})$. As for (2.15), $\int_\epsilon^\pi\int_\epsilon^\pi \csc\frac{1}{2}y\,O(y^2x^{-1}(x^2+y^2)^{-1})\,dx\,dy=O(\log^2\epsilon^{-1})$, and, using (2.4), with

$$\left\{\int_\epsilon^{\pi-\frac{1}{2}\epsilon}\int_{y+\frac{1}{2}\epsilon}^{\pi}+\int_{1.5\epsilon}^{\pi}\int_\epsilon^{y-\frac{1}{2}\epsilon}\right\}\csc\tfrac{1}{2}y\,y(x+y)\,|\,x-y\,|^{-1}(x^2+y^2)^{-1}dx\,dy = O(\log^2\epsilon^{-1}), \tag{2.16}$$

we see that both $\int_\epsilon^\pi \int_\epsilon^\pi \csc \tfrac{1}{2}y \mid I_{12} \mid dx\,dy$ and $\int_\epsilon^\pi \int_\epsilon^\pi \mid S_{11} \mid dx\,dy$ are $O(\log^2 \epsilon^{-1})$ for all $\mid x-y \mid \geq \tfrac{1}{2}\epsilon$.

Before considering the sum $S_{12}$ of (2.12) we discuss that sum of (2.8) whose exponent is $-mx+ny$. In particular we need only consider the sum analogous to $S_{11}$, namely,

$$(2.17) \qquad T_{11} = \frac{e(y)}{e(y)-1} \sum_{0<m\leq a} e(-mx + y(R-m^2)^{\frac{1}{2}} - \tfrac{1}{2}y),$$

since sums corresponding to $S_2$ and $S_{12}$ may be treated by the same methods as used for these sums. Here although the function $2\pi f(u) = -ux + y(R-u^2)^{\frac{1}{2}}$ has a monotone derivative, it is not bounded by $\frac{1}{2}$, indeed $f'(0) = -x(2\pi)^{-1}$, $f'(a) = -(x+y)(2\pi)^{-1}$ so that for $(\epsilon \leq x \leq \pi-\epsilon, \epsilon \leq y \leq \pi-\epsilon)$, $\epsilon(2\pi)^{-1} \leq \mid f'(u) \mid \leq 1-\epsilon\pi^{-1}$, and hence the hypotheses of the van der Corput theorem do not hold. However the same method of proof is applicable. By the Euler summation formula [5]

$$(2.18) \quad (e(y)-1)e(-\tfrac{1}{2}y)T_{11} = \int_0^a e(-ux + y(R-u^2)^{\frac{1}{2}})\,du - e(a(-x+y))b(a) - \tfrac{1}{2}e(yR^{\frac{1}{2}}) + \int_0^a b(u)\,d(e(-ux + y(R-u^2)^{\frac{1}{2}})),$$

where $b(u) = u - [u] - \frac{1}{2} = O(1)$ has the series representation

$$-\sum_{\nu=1}^{\infty} \sin 2\pi\nu/\pi\nu \quad ([u] < u < [u]+1),$$

which has uniformly bounded partial sums. The two middle terms of the right side of (2.18) are $O(1)$, so that the integral of their absolute value, multiplied by $\csc \frac{1}{2}y$, over $(\epsilon \leq x \leq \pi-\epsilon, \epsilon \leq y \leq \pi-\epsilon)$ contributes $O(\log \epsilon^{-1})$. Now consider the term

$$(2.19) \quad \int_0^a \{e(2\pi f(u) + \nu u) - e(2\pi f(u) - \nu u)\} f'(u)\,du = (2\pi i)^{-1} \int_0^a f'(u)(f'(u)+\nu)^{-1} d(e(2\pi f(u) + \nu u)) - (2\pi i)^{-1} \int_0^a f'(u)(f'(u)-\nu)^{-1} d(e(2\pi f(u) - \nu u)).$$

The functions $g_1(u) = f'(u)(f'(u)+\nu)^{-1}$ and $g_2(u) = f'(u)(f'(u)-\nu)^{-1}$ are monotone and do not change sign in $(0, a)$. Also, $\mid g_1(u) \mid \leq \nu^{-1}$ for $\nu \geq 1$ and $\mid g_2(u) \mid \leq (\nu-1)^{-1}$ for $\nu \geq 2$, while for $\nu = 1$, $\mid g_2(0) \mid = \mid x(2\pi - x)^{-1} \mid \leq 1$, $\mid g_2(a) \mid = (x+y)(2\pi - x - y)^{-1} \leq \pi\epsilon^{-1}$. Using the second mean value theorem on (2.19), we thus get

$$(2.20)\qquad J_1 = \int_0^a b(u)\,d(e(-ux + y(R-u^2)^{\frac{1}{2}})) = O(1) + O((x+y)(2\pi - x - y)^{-1}),$$

so that

$$(2.21)\qquad \int_\epsilon^{\pi-\epsilon}\int_\epsilon^{\pi-\epsilon} \csc \tfrac{1}{2}y \mid J_1 \mid dx\,dy = O(\log \epsilon^{-1}) + O\Big(\int_\epsilon^{\pi-\epsilon} \log(2\pi - x - y)\,y^{-1}dy\Big]_{x=\epsilon}^{\pi-\epsilon}\Big) = O(\log^2 \epsilon^{-1}).$$

Similarly in the first integral of (2.18), since $f'(u)$ is monotone and different from zero in $(0, a)$, $(f'(u))^{-1}$ is monotone and bounded in $(0, a)$ and using the second mean value theorem we get

$$J_2 = \int_0^a e(2\pi f(u))\,du = -2\pi x^{-1}O(1) - 2\pi(x+y)^{-1}O(1).$$

so that

$$\int_\epsilon^{\pi-\epsilon}\int_\epsilon^{\pi-\epsilon} \csc \tfrac{1}{2}y \mid J_2 \mid dx\,dy = O(\log^2 \epsilon^{-1})$$

as before. Thus finally

$$\int_\epsilon^{\pi-\epsilon}\int_\epsilon^{\pi-\epsilon} \mid T_{11} \mid dx\,dy = O(\log^2 \epsilon^{-1}),$$

**2.3.** There now remains to consider the sum $S_{12}$ of (2.12). In this case we have been unsuccessful in using a direct analysis,—the difficulty apparently arising from the discontinuity of $[(R-m^2)^{\frac{1}{2}}]$; indeed the only property of the exponential function that we use is that of boundedness. Since $e(y[(R-m^2)^{\frac{1}{2}}]) - e(y(R-m^2)^{\frac{1}{2}} - \frac{1}{2}y) = yb((R-m^2)^{\frac{1}{2}})O(1)$,

$$(2.22)\qquad S_{12} = O\Big(y \csc \tfrac{1}{2}y \sum_{0<m<a} \mid b((R-m^2)^{\frac{1}{2}}) \mid\Big).$$

We now use the following theorem to be found in [3, Ch. XII]:

*Let $f(u)$ be a function with two continuous derivatives in $1 \leq u \leq P$, and $0 < f'(u) < 1$, $f''(u) > z^{-3}$. Then $\sum_{u=1}^{P} \{f(u)\} = \frac{1}{2}P + O(z^2)$, where* $\{f(u)\} = \min(f(u) - [f(u)], [f(u)] + 1 - f(u))$.

Taking $f(u) = 1 - (R-u^2)^{\frac{1}{2}}$, we find that both $f'(u)$ and $f''(u)$ are monotonic non-decreasing in $(0, a)$ and positive. Also,

$$f'(0) = 0 < f'(u) < f'(a) = 1, \qquad f''(u) > f''(0) = R^{-\frac{1}{2}}$$

and $z = R^{1/6}$. Thus all the hypotheses of the theorem are satisfied. Now

$\{\xi\} = \{\xi - 1\}$ and $\{\xi\} = \{-\xi\}$ so that $\{f(u)\} = \{(R - u^2)^{\frac{1}{2}}\}$. Also, $0 \geq b(\xi) = \{\xi\} - \frac{1}{2}$ for $[\xi] \leq \xi \leq [\xi] + \frac{1}{2}$, and $0 \leq b(\xi) = \frac{1}{2} - \{\xi\}$ for $[\xi] + \frac{1}{2} \leq \xi \leq [\xi] + 1$, that is $| b(\xi) | = \frac{1}{2} - \{\xi\}$. Hence

$$S_{12} = O(y \csc \tfrac{1}{2}y \sum_{0<m<a} (\tfrac{1}{2} - \{(R - m^2)^{\frac{1}{2}}\})) \tag{2.23}$$
$$= O(R^{1/3}),$$

and thus $\int_\epsilon^\pi \int_\epsilon^\pi | S_{12} | \, dx \, dy = O(R^{1/3})$. Furthermore this result is independent of the signs used in the term $e(+mx + ny)$, and the proof of Theorem 2.1 is complete upon choosing $\epsilon = R^{-\frac{1}{2}}$.

**2.4.** THEOREM 2.2. *If* $\sum (m^2 + n^2)^{1/3} | a_{mn} |^2 < \infty$, *then* $\lim_{R\to\infty} S_R$ *exists almost everywhere on* $C(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi)$, *where*

$$S_R(\alpha, \beta) = \sum_{\nu \leq R} \sum_{\nu = m^2+n^2} a_{mn} e(m\alpha + n\beta).$$

Following a classical method of proof of Fourier and orthogonal series [4], we first show by means of Theorem 2.1 that $S_R$ is $O(R^{1/6})$ a. e. (almost everywhere) on $C$, then that the convergence of $\sum (m^2 + n^2)^{1/3} | a_{mn} |^2$ implies the convergence of a certain subsequence of $\{S_R\}$ and finally that these two results imply the convergence of the sequence $\{S_R\}$ a. e. on $C$.

LEMMA 2.1. *If* $\sum | a_{mn} |^2 < \infty$, *then* $S_R(\alpha, \beta) = O(R^{1/6})$ *a. e. on* $C$.

*Proof.* Since $\sum | a_{mn} |^2 < \infty$, by the Riesz-Fischer Theorem extended to more variables there exists a real Lebesgue square integrable function $f$, whose Fourier coefficients, $(4\pi)^{-2} \int\int_c f(x, y) e(-(mx + ny)) \, dx \, dy$, equal $a_{mn}$.

Now following a proof to be found in [4], we set $v_R = \max_{0<\nu\leq R} S_\nu(P) \nu^{-1/6}$ $= t^{-1/6} S_t(P)$ $(R \neq 0)$, $v_0 = a_{00}$, where $P = (\alpha, \beta)$ and $t = t(P)$. The sequence $\{v_R\}$ $(R \neq 0)$ is monotone non-decreasing and we show that $I_R = \int\int_c v_R dx \, dy = O(1)$. The integral

$$I = \int\int_c f(P') dx' dy' \int\int_c t^{-1/6} K_t(P - P') \, dx \, dy,$$

where $P - P' = (x - x', y - y')$, and the Schwarz inequality, setting $B = \int\int f^2 dx \, dy$, we get $I^2_R \leq B \int\int | \int\int t^{-1/6} K_t(P - P') \, dx \, dy |^2 \, dx' dy'$. Evaluating the latter integral [cf. 8, p. 253], we have

$$\int\int dx' dy' \int\int t^{-1/6} K_t(P - P') \, dx \, dy \int\int t_0^{-1/6} K_{t_0}(P_0 - P') \, dx_0 dy_0$$
$$= \int\int\int\int t^{-1/6} t_0^{-1/6} K_T(P - P') \, dx \, dy \, dx_0 dy_0 \quad (T = \min(t, t_0)),$$

since by the orthonormal relations $\int\int K_t(P-P')K_{t_0}(P_0-P')dx'dy' = K_T(P-P_0)$. Consequently

$$I^2_R \leq B\int\int\int\int (tt_0)^{-1/6}\,|K_T(P-P_0)|\,dx\,dy\,dx_0dy_0$$
$$\leq B\{\int\int t^{-1/3}dx\,dy\int\int |K_t(P-P_0)|\,dx_0dy_0$$
$$+\int\int t_0^{-1/3}dx_0dy_0\int\int |K_{t_0}(P-P_0)|\,dx\,dy,$$

which by Theorem 2.1 is $O(1)$. Therefore $\lim_{R\to\infty} v_R(P)$ exists a. e. and the sequence $\{S_R(P)/R^{1/6}\}$ is bounded above a. e. Similarly the sequence is bounded below a. e. and the lemma is proved.

LEMMA 2.2. *If the series* $\sum \alpha_\nu (\alpha_\nu > 0)$ *converges, then there exists a monotone sequence* $\{p_\nu\}$, $p_\nu \to \infty$, *such that* $\sum p_\nu \alpha_\nu < \infty$.

This lemma is well-known.

LEMMA 2.3. *If* $\sum |a_{mn}|^2 (m^2+n^2)^{1/3} < \infty$, *then* $\{S_{\nu_k}\}$ *approaches a limit a. e. on C, where* $\{\nu_k\}$ *is a subsequence of the sequence* $\{R\}$ *of* (1.3) *satisfying* $k^3 \leq \nu_k < (k+1)^3$.

*Proof.* By the orthonormal properties we have

$$\int\int |f(P)-S_{\nu_k}|^2 dx\,dy = \sum_{\nu<\nu_k}\sum_{\nu=m^2+n^2} |a_{mn}|^2 = r_{\nu_k}.$$

The series $\sum_{k=0}^{\infty} r_{\nu_k}$ is convergent, for by Abel's transformation we get

$$\sum_{k=0}^{\infty} r_{\nu_k} = \sum_{k=0}^{\infty} k(r_{\nu_k} - r_{\nu_{k+1}}) + \lim_{k\to\infty} k r_{\nu_{k+1}}$$
$$= \sum_{k=0}^{\infty} k \sum_{\nu_k<\nu\leq\nu_{k+1}} \sum_{\nu=m^2+n^2} |a_{mn}|^2 + \lim_{k\to\infty} k \sum_{\nu>\nu_{k+1}} \sum_{\nu=m^2+n^2} |a_{mn}|^2.$$

But $k \leq \nu_k^{1/3} < \nu^{1/3} = (m^2+n^2)^{1/3}$. Hence the right hand side is less than or equal to

$$\sum_{k=0}^{\infty} \sum_{\nu_k<\nu\leq\nu_{k+1}} \sum_{\nu=m^2+n^2} (m^2+n^2)^{1/3}\,|a_{mn}|^2$$
$$+ \lim_{k\to\infty} \sum_{\nu>\nu_{k+1}} \sum_{\nu=m^2+n^2} (m^2+n^2)^{1/3}\,|a_{mn}|^2,$$

which equals $\sum |a_{mn}|^2 (m^2+n^2)^{1/3}$, since the convergence is absolute. Thus $\sum_{k=0}^{\infty} \int\int |f(P)-S_{\nu_k}|^2 dx\,dy < \infty$; consequently $\sum_{k=0}^{\infty} |f(P)-S_{\nu_k}|^2 < \infty$ a. e. on $C$ and $\lim_{k\to\infty} S_{\nu_k} = f(P)$ a. e. on $C$.

*Proof of Theorem* 2.2 [4]. From Lemma 2.3 it remains to prove that $\lim_{k\to\infty}(S_R(P) - S_{\nu_k}(P)) = 0$ a. e. for $\nu < R < \nu_{k+1}$, where from Lemma 2.3 $k \leq \nu_k^{1/3} < k+1$. By Lemma 2.2 there exists a monotone sequence $\{p_\nu\}$ with $p_\nu \to \infty$, such that $\sum p_\nu^2 \nu^{1/3} |\, a_{mn}\,|^2 < \infty$ $(\nu = m^2+n^2)$. Setting $b_{mn} = p_\nu \nu^{1/6} a_{mn}$, $c_\nu = p_\nu \nu^{1/6}$, $\Phi_\mu = \sum_{m^2+n^2=\mu} b_{mn} e(m\alpha + n\beta)$ we have by Abel's transformation for $k > 0$

$$S_R - S_{\nu_k} = \sum_{\nu_k < \nu \leq R} \Big( \sum_{\mu=0}^{\nu} \Phi_\mu (c_\nu^{-1} - c_{\nu+1}^{-1}) - c_{\nu_{k+1}}^{-1} \sum_{0 \leq \nu \leq \nu_k} \Phi_\nu + c_{R+1}^{-1} \sum_{0 \leq \nu \leq R} \Phi_\nu .$$

Since $\sum |\, b_{mn}\,|^2 < \infty$, $b_{mn}$ are the Fourier coefficients of a real Lebesgue square integrable function. Hence by Lemma 2.1 $\sum_{0 \leq \nu \leq \nu_k} \Phi_\nu = O(\nu_k^{1/6})$ and therefore

$$|\, S_R - S_{\nu_k}\,| = O(R^{1/6})\,(c_{\nu_{k+1}}^{-1} - c_{R+1}^{-1}) + O(p_{\nu_{k+1}}^{-1}) + O(p_{R+1}^{-1}).$$

Now $R\nu_k^{-1} \leq \nu_{k+1}\nu_k^{-1} \leq (k+2)^3 k^{-3} = O(1)$, so that $|\, S_R - S_{\nu_k}\,| = O(p_{\nu_k}^{-1})$, which $\to \infty$ as $k \to \infty$. Hence the theorem is proved.

**3. The case $q = 3$.** The result corresponding to Theorem 2.1 is

THEOREM 3.1. *The Lebesgue function $L_R(\alpha, \beta, \gamma)$ is $O(R^{2/3}(\log R)^{5/3})$.*

*Proof.* As in the proof of Theorem 2.1 we subdivide the cube $C(-\pi \leq x \leq \pi, -\pi \leq y \leq \pi, -\pi \leq z \leq \pi)$ by drawing planes parallel to the $y$- and $z$-axes through the points $-\pi+\epsilon, -\epsilon, 0, \epsilon, \pi-\epsilon$, where $\epsilon$ is an arbitrary small positive number. Using the bound $K_R(x,y,z) = O(R^{3/2})$, we get that the integral of $|\, K_R(x,y,z)\,|$ over parallelepipeds of dimension $\epsilon \times \epsilon \times 2\pi$ is $O(\pi\epsilon^2 R^{3/2})$. For parallelepipeds of dimension $\epsilon \times \pi - 2\epsilon \times 2\pi$ analogously to (2.2), we use

$$(3.2)\quad K_R(x,y,z) = \frac{1}{e(z)-1} \sum_{m^2+n^2 \leq R} e(mx+ny) \times \{e(([R-m^2-n^2)^{\frac{1}{2}}]+1)z) - e(-z[(R-m^2-n^2)^{\frac{1}{2}}])\} = O(R \csc \tfrac{1}{2}z),$$

if $|\, z\,| \geq \epsilon$, so that, for example,

$$\int_{-\pi}^{\pi} dx \int_{\epsilon}^{\pi} dz \int_0^{\epsilon} |\, K_R(x,y,z)\,|\, dy = O(\epsilon R \log \epsilon^{-1}).$$

Similarly as in (2.4),

$$(3.3)\quad \int_{-\pi}^{\pi} dx \int_{\epsilon}^{\pi-\epsilon} dz \int_{\alpha z - \frac{1}{2}\epsilon}^{\alpha z + \frac{1}{2}\alpha} |\, K_R(x,y,z)\,|\, dy = O(\epsilon R \log \epsilon^{-1}),\ (\alpha \text{ a constant}).$$

In order to discuss the sum $\sum_{m^2+n^2+p^2\leq R}$ for parallelepipeds of dimension $\pi-2\epsilon \times \pi-2\epsilon \times 2\pi$, following Hua [3], we subdivide the sphere $m^2+n^2+p^2 \leq R$ by planes $m=0$, $n=0$, $p=0$, $m=n$, $m=p$ and $n=p$. On the plane $n=p$, for example, we have

$$(3.4)\quad Q_1 = \sum_{m^2+2n^2\leq R} e(mx+(y+z)n)$$
$$= \frac{1}{e(y+z)-1} \sum_{m=-[R^{1/2}]}^{[R^{1/2}]} e(mx)\{e([(\tfrac{1}{2}(R-m^2))^{\frac{1}{2}}]+1)(y+z) - e(-[(\tfrac{1}{2}(R-m^2))^{\frac{1}{2}}](y+z))\}$$
$$= O(R^{\frac{1}{2}}\csc \tfrac{1}{2}(y+z)), \qquad (y+z \neq 0 \text{ or } \pm 2\pi),$$

so that as in (2.6) $\int_{-\pi}^{\pi} dx \int_{\epsilon}^{\pi-\epsilon}\int_{\epsilon}^{\pi-\epsilon} |Q_1|\,dy\,dz = O(R^{\frac{1}{2}}\log \epsilon^{-1})$. On the other hand over $(-\pi \leq x \leq \pi, \epsilon \leq y \leq \pi-\epsilon, -\pi+\epsilon \leq z \leq -\epsilon)$ we use the procedure of (3.3) with $\alpha=-1$ and (2.7) to obtain the same bound. Similar bounds are obtained on the planes $m=0$, $n=0$, $m=n$ and $m=p$.

The remaining sums to be considered in $\sum_{m^2+n^2+p^2\leq R}$ are all of the form

$$(3.5)\quad \sum_{m^2+n^2+p^2\leq R} e(\pm mx \pm ny \pm pz)$$
$$(-\pi \leq x \leq \pi, \epsilon \leq y \leq \pi-\epsilon, \epsilon \leq z \leq \pi-\epsilon) = P_\epsilon, \text{ where } 0<m<n<p$$

or this sum with $m$, $n$ and $p$ permuted. In considering (3.5) the discontinuous bound $O(|y-\alpha_m z|^{-1})$, where $\alpha_m = m(R-2m^2)^{-\frac{1}{2}}$ $(0<m\leq(R/3)^{\frac{1}{2}})$, is encountered. This bound cannot be integrated over the domain

$$(3.6)\quad D_{m\epsilon} = (-\pi \leq x \leq \pi, \epsilon \leq z \leq \pi-\epsilon, \alpha_m z - \tfrac{1}{2}\epsilon \leq y \leq \alpha_m z + \tfrac{1}{2}\epsilon),$$

but as for the case $q=2$, where the bound $O(|y-z|^{-1})$ occurred, we replace $O(|y-\alpha_m z|^{-1})$ over $D_{m\epsilon}$ by $|K_R(x,y,z)|$ plus the sum of all other bounds used in considering (3.5) and notice that all of these bounds may be integrated over $D_{m\epsilon}$ to give the same bound or an even sharper result than the bound obtained over $P_\epsilon$. Of course, since $\alpha_m$ depends on $m$ it is necessary to integrate over $D_{m\epsilon}$ before summing with respect to $m$, which will be seen to be possible in all cases.

Defining $G(R)$ as the set $(0<m\leq(R/3)^{\frac{1}{2}},\ m<n<(\frac{1}{2}(R-m^2))^{\frac{1}{2}})$, we consider as in (2.9)

$$(3.7)\quad S = \sum_{(m,n)\in G(R)} \sum_{n<p<(R-m^2-n^2)^{1/2}} e(mx+ny+pz)$$
$$= \frac{e(z)}{e(z)-1} \sum_{(m,n)\in G(R)} \{e(mx+ny+z[(R-m^2-n^2)^{\frac{1}{2}}]) - e(mx+(y+z)n)\}$$

$= S_1 + S_2$. The sum $S_2$ may be treated the same way as in (3.4) and hence we get $\iiint_{P_\epsilon} | S_2 | \, dx\, dy\, dz = O(R^{\frac{1}{2}} \log^2 \epsilon^{-1})$ (cf. (2.10)). We can use the same methods on the sum $S_1$ as were used on the corresponding sum for $q = 2$, that is, we consider

$$(3.8) \quad S_{11} = \frac{e(z)}{e(z) - 1} \sum_{(m,n) \in G(R)} e(mx + ny + z(R - m^2 - n^2)^{\frac{1}{2}} - \tfrac{1}{2}z)$$

and

$$(3.9) \quad S_{12} = \frac{e(z)}{e(z) - 1} \sum_{(m,n) \in G(R)} \{e(mx + ny + z[(R - m^2 - n^2)^{\frac{1}{2}}]) - e(mx + ny + z(R - m^2 - n^2)^{\frac{1}{2}} - \tfrac{1}{2}z)\},$$

where $S_1 = S_{11} + S_{12}$.

In $S_{11}$ the function $2\pi f(u) = yu + z(R - m^2 - u^2)^{\frac{1}{2}}$ $(m < u < R_m, R_m = (\frac{1}{2}(R - m^2))^{\frac{1}{2}})$ satisfies the hypotheses of the van der Corput theorem for $(0 \leq y \leq \pi, 0 \leq z \leq \pi; 0 \leq m \leq R_0 = (R/3)^{\frac{1}{2}})$. Consequently

$$(3.10) \quad e(-\tfrac{1}{2}z)(e(z) - 1) S_{11} = \sum_{0 < m \leq R_0} \{\int_m^{R_m} e(mx + uy + z(R - m^2 - u^2)^{\frac{1}{2}})\, du + O(1)\}.$$
$$= \sum_{0 < m \leq R_0} e(mx) I_m + O(R^{\frac{1}{2}}).$$

Using the same procedure as for $q = 2$, we show that

$$(3.11) \quad \sum_{0 < m \leq R_0} \iiint_{P_\epsilon - D_{m\epsilon}} \csc \tfrac{1}{2}z \, | I_m | \, dx\, dy\, dz = O(R^{\frac{1}{2}} \log^2 \epsilon^{-1}),$$

where $P_\epsilon - D_{m\epsilon}$ is the set of points belonging to $P_\epsilon$ but not to $D_{m\epsilon}$ and $\alpha_m = m(R - 2m^2)^{-\frac{1}{2}}$.

*Proof.* Setting $rv = wy + z(R - m^2 - u^2)^{\frac{1}{2}}$, where $r = (y^2 + z^2)^{\frac{1}{2}}$ and $ru = yv - z(R - m^2 - v^2)^{\frac{1}{2}}$, we get

$$I_m = \frac{1}{r} \int_{v_1}^{v_2} (y + \frac{zv}{(R - m^2 - v^2)^{\frac{1}{2}}}) e(rv)\, dv = I_{m1} + I_{m2},$$

where $v_1 = r^{-1}(my + z(R - 2m^2)^{\frac{1}{2}})$ and $v_2 = r^{-1}(y + z)R_m$. Now $I_{m1} = O(yr^{-2})$ and as in (2.15)

$$\sum_{0 < m \leq R_0} \iiint_{P_\epsilon - D_{m\epsilon}} \csc \tfrac{1}{2}z \, | I_{m1} | \, dx\, dy\, dz \leq \sum_{0 < m \leq R_0} \iiint_{P_\epsilon} \csc \tfrac{1}{2}z \, | I_{m1} | \, dx\, dy\, dz$$
$$= O(R^{\frac{1}{2}} \log^2 \epsilon^{-1}).$$

The function $g(v) = v(R - m^2 - v^2)^{-\frac{1}{2}}$ of $I_{m2}$ is monotone and hence by the second mean value theorem

$$I_{m2} = zr^{-2}g(v_1)O(1) + zr^{-2}g(v_2)O(1),$$

where

$$g(v_1) = (my + z(R - 2m^2)^{\frac{1}{2}})(mz - y(R - 2m^2)^{\frac{1}{2}})^{-1}$$

and

$$g(v_2) = (y + z)|\, y - z\,|^{-1},$$

which equals $g(v_1)$ for $m = R_0$. As a function of $y$ and $z$, $g(v_1)$ is discontinuous at $y = \alpha_m z$, where $R^{-\frac{1}{2}} < \alpha_m \leq 1$. Moreover for all such $\alpha_m$ the line $y = \alpha_m z$ crosses the square $(\epsilon \leq y \leq \pi - \epsilon, \epsilon \leq z \leq \pi - \epsilon)$ if $\epsilon$ is of the order $O(R^{-\frac{1}{2}})$ (cf. later). To avoid the discontinuity at $y = \alpha_m z$ we use (3.3) with $\alpha = \alpha_m$ and notice that all the other bounds are either $O(R^{\frac{1}{2}} \log^2 \epsilon^{-1})$ or $O(R^{5/6}\epsilon)$ (cf. (3.20)) over the domain $\sum_{0<m\leq R_0} D_{m\epsilon}$. Over $P_\epsilon - D_{m\epsilon}$, evaluating as in (2.16) we get

$$\iiint_{P\epsilon - D_{m\epsilon}} zr^{-2} \csc \tfrac{1}{2}z(\alpha_m y + z)|\, \alpha_m z - y\,|^{-1}\, dx\, dy\, dz = O(\log^2 \epsilon^{-1}),$$

which proves (3.11) upon summing with respect to $m$.

As in the case $q = 2$ and since our proofs are independent of $x$, in addition to $S$ of (3.7) it is sufficient to discuss that sum whose exponent is $mx + ny - pz$. In particular we need only consider

$$T_1 = \frac{e(-\frac{1}{2}z)}{e(-z) - 1} \sum_{(m,n)\,\epsilon\, G(R)} e(mx + ny - z(R - m^2 - n^2)^{\frac{1}{2}}), \qquad (x, y, z)\ \epsilon\ P_\epsilon \tag{3.12}$$

(cf. (2.17)). As in the case $q = 2$ the van der Corput theorem fails for this sum but the method of proof is applicable. Hence by Euler's formula

$$\begin{aligned} T_1 = \frac{e(-\frac{1}{2}z)}{e(-z) - 1} \sum_{0<m\leq R_0} \Big\{ & \int_m^{R_m} e(mx + uy - z(R - m^2 - u^2)^{\frac{1}{2}})\, du \\ & - e(mx + R_m(y - z))\, b(R_m) - \tfrac{1}{2} e(m(x + y) - z(R - m^2)^{\frac{1}{2}}) \\ & + \int_m^{R_m} b(u)\, d(e(mx + uy - z(R - m^2 - u^2)^{\frac{1}{2}}))\Big\} \\ = \frac{e(-\frac{1}{2}z)}{e(-z) - 1} & (T_{11} + T_{12} + T_{13} + T_{14}). \end{aligned} \tag{3.13}$$

In this case the function $2\pi f(u) = yu - z(R - m^2 - u^2)^{\frac{1}{2}}$ has a bounded monotone derivative which does not change sign in $(m, R_m)$; indeed

$$(2\pi)^{-1}\epsilon \leq f'(m) = (2\pi)^{-1}(y + \alpha_m z) \leq f'(u) \leq f'(R_m) = (2\pi)^{-1}(y + z),$$

so that $|f'(u)| \leq 1 - \epsilon(\pi)^{-1}$. Thus $1/f'(u)$ is bounded and monotone in the interval $(m, R_m)$ and by the second mean value theorem

$$J_{1m} = \int_m^{R_m} (2\pi f'(u))^{-1} d(e(2\pi f(u))) = O(y^{-1}) + O((y+z)^{-1}),$$

since $f'(m) \geq (2\pi)^{-1}y$, so that

$$\iiint_{P\epsilon} \csc \tfrac{1}{2}z \,|\, \sum_{0<m\leq R_0} e(mx) J_{1m} \,|\, dx\, dy\, dz = O(R^{\frac{1}{2}} \log^2 \epsilon^{-1}).$$

Also, $T_{12}$ and $T_{13}$ in (3.13) are $O(R^{\frac{1}{2}})$. On $T_{14}$ we use the same procedure as for (2.19) since the corresponding functions $g_1(u) = f'(u)(f'(u)+\nu)^{-1}$ and $g_2(u) = f'(u)(f'(u)-\nu)^{-1}$ are also monotone and bounded and do not change sign in $(m, R_m)$. The only difference is that

$$g_2(m) = (y + \alpha_m z)(y + \alpha_m z - 2\pi\nu)^{-1},$$

but

$$|g_2(m)| \leq (y+z)(2\pi\nu - y - z)^{-1},$$

which is the value of $g_2(R_m)$ and similar to the case $q = 2$. Thus

$$\iiint_{P\epsilon} |T_1|\, dx\, dy\, dz = O(R^{\frac{1}{2}} \log^2 \epsilon^{-1}).$$

Now consider $S_{12}$ of (3.9). Using the same argument as for the case $q = 2$,

$$e(z[(R - m^2 - n^2)^{\frac{1}{2}}]) - e(z((R - m^2 - n^2)^{\frac{1}{2}} - \tfrac{1}{2})) = zb((R - m^2 - n^2)^{\frac{1}{2}})O(1),$$

so that

$$S_{12} = O(z \csc \tfrac{1}{2}z \sum_{(m,n)\in G(R)} |b((R - m^2 - n^2)^{\frac{1}{2}})|) \tag{3.14}$$

$$= O(z \csc \tfrac{1}{2}z \sum_{(m,n)\in G(R)} (\tfrac{1}{2} - \{(R - m^2 - n^2)^{\frac{1}{2}}\})).$$

But by a theorem due to Hua[3]

$$\sum_{(m,n)\in G(R)} \{(R - m^2 - n^2)^{\frac{1}{2}}\} = \tfrac{1}{2}P + O(R^{2/3}(\log R)^{5/3}), \tag{3.19}$$

where $P = \sum_{(m,n)\in G(R)} 1$. Consequently

$$\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} |S_{12}|\, dx\, dy\, dz = O(R^{2/3}(\log R)^{5/3}).$$

In connection with (3.11) we must show that

$$\sum_{(m,n)\in G(R)} \iiint_{D_{m\epsilon}} |b((R - m^2 - n^2)^{\frac{1}{2}})|\, dx\, dy\, dz$$

is sufficiently small. For fixed $m$ the function $f(u) = 1 - (R - m^2 - n^2)^{\frac{1}{2}}$ satisfies the hypotheses of the theorem quoted on p. 8 where the $z$ of the theorem may be taken to be $R^{1/6}$ again. Thus by subtraction we get

$$\sum_{m \le n < R_m} \{(R - m^2 - n^2)^{\frac{1}{2}}\} = \tfrac{1}{2} \sum_{m \le n < R_m} 1 + O(R^{1/3}),$$

or

$$\sum_{m < n < R_m} | b((R - m^2 - n^2)^{\frac{1}{2}}) | = O(R^{1/3}),$$

so that

$$(3.20) \qquad \sum_{0 < m \le R_0} \iiint_{D_{m\epsilon}} \sum_{m < n \le R_m} | b((R - m^2 - n^2)^{\frac{1}{2}}) | \, dx \, dy \, dz = O(R^{5/6}\epsilon).$$

Since these proofs are independent of the signs in front of the $x$, $y$ and $z$ terms in $e(\pm mx \pm ny \pm pz)$, the proof of Theorem 3.1 is complete if we take $\epsilon = R^{-\frac{1}{2}}$. Since the sequence $\{(m^2 + n^2 + p^2)^{2/3}(\log(m^2 + n^2 + p^2))^{5/3}\}$ is monotonic increasing,, the proof of Theorem 2.2 generalizes without difficulty so that we have

THEOREM 3.2. *If the series*

$$\Sigma(m^2 + n^2 + p^2)^{2/3}(\log(m^2 + n^2 + p^2))^{5/3} \, | \, a_{mnp} \, |^2$$

*is convergent, then* $\lim_{R \to \infty} S_R$ *exists a. e. on* $(-\pi \le x \le \pi, \ -\pi \le y \le \pi, \ -\pi \le z \le \pi)$.

**4.** *The general case.* The following theorem is readily proved:

THEOREM 4.1. *The Lebesgue function* $L_R(\alpha_1, \cdots, \alpha_q)$ *is* $O(R^{\frac{1}{2}q})$.

*Proof.* The proof depends only on the boundedness and orthogonality properties of the exponential system. By the Schwarz inequality

$$(4.1) \qquad L^2{}_R(\alpha_1, \cdots, \alpha_q) = \{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} | K_R(P - Q) | \, dA\}^2$$

$$= O(\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} | K_R(P - Q) |^2 \, dA),$$

where $P = (\alpha_1, \cdots, \alpha_q)$, $Q = (x_1, \cdots, x_q)$ and $dA = dx_1 \cdots dx_q$. By the orthonormal properties (1.1)

$$\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} | K_R(P - Q) |^2 \, dA = \sum_{\nu \le R} 1, \quad \text{where } \nu = m^2{}_1 + \cdots + m^2{}_q.$$

But the right hand side of this equation is the number of lattice points in the hypersphere $m^2{}_1 + \cdots + m^2{}_q \le R$, which is $O(R^{\frac{1}{2}q})$ [6]. Hence the conclusion of the theorem follows.

Using Theorem 4.1, it is easy to prove by the method of paragraph 2.4

THEOREM 4.2. *If* $\sum(m^2_1 + \cdots + m^2_q)^{\frac{1}{2}q} \mid a_{m_1 \ldots m_q} \mid^2 < \infty$, *then* $\lim_{R\to\infty} S_R$ *exists a. e. on* $(-\pi \leq x_k \leq \pi,\ k = 1, \cdots, q)$.

We remark that it would be impossible to improve this theorem by the methods used in Theorems 2.1 and 3.1 inasmuch as present results give the number of lattice points in the hypersphere

$$m^2_1 + \cdots + m^2_q \leq R \text{ as } \pi^{\frac{1}{2}q}(\Gamma(\tfrac{1}{2}q + 1))^{-1}R^{\frac{1}{2}q} + O(R^{\frac{1}{2}q-1})$$

for $q \geq 5$ and as $\frac{1}{2}\pi^2R^2 + O(R^{1+\epsilon})$ for $q = 4$ [7].

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# ORDER IN OPERATOR ALGEBRAS.*

By S. SHERMAN.

A partial order may be introduced among the hermitian elements of a $C^*$ algebra $\mathcal{A}$ by letting $U \geqq 0$ mean that $U$ is positive semi-definite. It is well known [7] [1, Lemma 1] that if a $C^*$ algebra is commutative, the ordering (always the one given above) is a lattice ordering. We show here that if the hermitian elements of a self-adjoint algebra (with or without unit, no topological assumptions) of bounded operators on Hilbert space form a lattice, then the algebra is commutative. By example it is shown that lattice ordered subspaces of hermitian elements of $\mathcal{A}$ are not necessarily (commutative) algebras.

Let $\mathcal{A}$ be a $C^*$ algebra [6, page 75], i. e. a uniformly closed self-adjoint complex algebra of bounded operators on a complex Hilbert space $H$. Thus if $U$, $V \in \mathcal{A}$ and $\alpha$ is a complex number, then $UV$, $U+V$, $V^*$ (the adjoint of $V$) and $\alpha V$ are elements of $\mathcal{A}$. We define

$$\| U \| = \sup\{ (Ux, Ux)^{\frac{1}{2}} \mid (x, x) = 1, x \in H\}$$

and the requirement of uniform closure means that $\mathcal{A}$ is complete with respect to the norm $\| U \|$. Let $\mathcal{S} = \{U \mid U \in \mathcal{A}, U^* = U\}$ be the set of hermitian elements of $\mathcal{A}$. We say that $U \geqq 0$ if $(Ux, x) \geqq 0$ for each $x \in H$. This implies that $U \in \mathcal{S}$. Let $U > 0$ mean $U \geqq 0$ and $U \neq 0$.

We wish to investigate consequences of the assumption that $\mathcal{S}$ is a lattice, i. e. for every $U \in \mathcal{S}$, there is a $U_+ \geqq 0$ such that $U_+ - U \geqq 0$ and $U' \geqq U_+$ for any $U' \in \mathcal{S}$ that satisfies the two conditions $U' \geqq 0$ and $U' - U \geqq 0$. Denote by $U_-$ the element $(-U)_+$ and by $| U |$ the element $U_+ + U_-$. Thus $U = U_+ - U_-$ and $| U | = U_+ + U_-$. In order not to obscure the essential lines of the proof with details we first handle the case with unit, i. e. the case where $\mathcal{A}$ contains the identity operator $I$.

THEOREM 1. *If $\mathcal{S}$, the set of hermitian elements of $\mathcal{A}$, which is a $C^*$ algebra with unit, is lattice ordered, then $\mathcal{A}$ is commutative.*

*Proof.* $\mathcal{S}$ satisfies Axioms I-V of [2], if the $u$, $\| U \|_u$, and $E_u$ of [2] are $I$, $\| U \|$, and $\mathcal{S}$ respectively. A linear functional $\omega \mid \mathcal{S}$ is called positive ($\omega \geqq 0$) if $U \geqq 0$ implies $\omega(U) \geqq 0$. As in [2] we define $\| \omega \| = \omega(I)$.

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Let $\Omega$ be the set of all positive functionals having norm 1, i. e. $\omega \geqq 0$ and $\omega(I) = 1$. Let $\Pi$ be the set of extreme points of $\Omega$. Note: when the domain of $\omega \mid \mathcal{S} \in \Omega$ is extended in a natural way so that we get $\bar{\omega} \mid \mathcal{A}$, Segal refers to $\bar{\omega}$ as a state and if $\omega \mid \mathcal{S} \in \Pi$, then $\bar{\omega} \mid \mathcal{A}$ is called by Segal a pure state. We use this terminology. [2] refers to $\Omega$ and $\Pi$ as $H_u$ and $S_u$ respectively. As in Theorem 3 of [2] we associate with each $U \in \mathcal{S}$ a function $\phi_U \in C(\Pi, R)$, where $\phi_U(\omega) = \omega(U)$ for each $\omega \in \Pi$ and $C(\Pi, R)$ is the collection of all continuous real-valued functions on the compact Hausdorff space $\Pi$ (using $w^*$ topology). According to [2] the one-one linear correspondence $U \leftrightarrow \phi_U$ possesses the following properties:

1) $$U \leqq V \Longleftrightarrow \phi_U(\omega) \leqq \phi_V(\omega) \text{ for each } \omega \in \Pi,$$

2) $$\| U \| = \sup\{|\phi_U(\omega)| \mid \omega \in \Pi\} \qquad (= df \, \|\phi_U\|),$$

3) $$\phi_I(\omega) = 1 \text{ for each } \omega \in \Pi,$$

4) $$\{\phi_U \mid U \in \mathcal{S}\} = C(\Pi, R).$$

Let $\mathcal{J}_0 = \{U \mid \bar{\omega}_0(U) = 0\}$, where $\bar{\omega}_0 \mid \mathcal{S} = \omega_0 \in \Pi$. We note that $\mathcal{J}_0$ is a closed complex linear subspace of $\mathcal{A}$ and we wish to show that $\mathcal{J}_0$ is a two-sided ideal of $\mathcal{A}$. For each $V \in \mathcal{A}$, $V = V_R + iV_I$ where $V_R = (V + V^*)2^{-1}$, $V_I = (V - V^*)(2i)^{-1}$, $V_R \in \mathcal{S}$, and $V_I \in \mathcal{S}$. (We note in passing that if we let $\phi_V(\omega) = \phi_{V_R}(\omega) + i\phi_{V_I}(\omega)$, then $\{\phi_V \mid V \in \mathcal{A}\} = C(\Pi, C)$, the collection of all continuous complex-valued functions on $\Pi$.) In particular $U \in \mathcal{J}_0 \Longrightarrow U_R \in \mathcal{S} \cap \mathcal{J}_0$, $U_I \in \mathcal{S} \cap \mathcal{J}_0$. Let $\mathcal{S}_+ = \{V \mid \phi_V(\omega) \geqq 0 \text{ for each } \omega \in \Pi\}$. For any $V \in \mathcal{S} \cap \mathcal{J}_0$, $V = V_+ - V_-$, where $V_+ \in \mathcal{S}_+ \cap \mathcal{J}_0$, $V_- \in \mathcal{S}_+ \cap \mathcal{J}_0$, $\phi_{V_+}(\omega) = \sup\{\omega(V), 0\}$ and $\phi_{V_-}(\omega) = \sup\{-\omega(V), 0\}$. What we wish to show is that $U \in \mathcal{J}_0$, $V \in \mathcal{A} \Longrightarrow UV \in \mathcal{J}_0$ and $VU \in \mathcal{J}_0$. The above indicates that it is sufficient to show that $U \in \mathcal{J}_0 \cap \mathcal{S}_+$, $V \in \mathcal{A} \Longrightarrow UV \in \mathcal{J}_0$ and $VU \in \mathcal{J}_0$.

Suppose $U \in \mathcal{J}_0 \cap \mathcal{S}_+$. Then by [1, page 201, Corollary 2] there exists a $W \in \mathcal{S}$ such that $W \geqq 0$ and $W^2 = U$. Obviously $\omega_0(VU) = \omega_0(ZW)$, where $Z = VW$, and by the Cauchy-Schwarz inequality

$$|\omega_0(ZW)| \leq (\omega_0(W^2))^{\frac{1}{2}}(\omega_0(ZZ^*))^{\frac{1}{2}} = 0.$$

From this it follows that $VU \in \mathcal{J}_0$ and analogously $UV \in \mathcal{J}_0$. Thus $\mathcal{J}_0$ is a two-sided ideal in $\mathcal{A}$. Since, for each $V \in \mathcal{A}$, $\bar{\omega}_0(V - \bar{\omega}_0(V)I) = 0$ and so $V - \bar{\omega}_0(V)I \in \mathcal{J}_0$, we see that $\bar{\omega}_0 : V \to \bar{\omega}_0(V)$ defines a homomorphism from $\mathcal{A}$ to the complex numbers. Thus $\bar{\omega}_0(UV) = \bar{\omega}_0(U)\bar{\omega}_0(V) = \bar{\omega}_0(VU)$ and $\phi_{UV}(\omega_0) = \phi_{VU}(\omega_0)$ for each $\omega_0 \in \Pi$. Since $U \leftrightarrow \phi_U$ is one-one, $UV = VU$ for each $U$, $V \in \mathcal{A}$, which proves our theorem. Incidentally since $\mathcal{A}$ is

commutative, $\mathfrak{S}$ is a real commutative algebra. Note that the arguments would be valid also for a $B^*$ algebra with unit [5, page 528], where with an appropriate definition of $\geqq$ we assume $U_1 \geqq U_2 \geqq 0 \Longrightarrow | U_1 | \geqq | U_2 | \geqq 0$.

The proof of the result analogous to Theorem 1 but without the assumption of a unit must contend with the possible distinction between the suprema (relative to imbedding ring) of a pair of elements $\{U_1, U_2\}$, imbedded in two rings one containing the other, $\{U_1, U_2\} \subset R_1 \subset R_2$. Furthermore the representation of $\mathfrak{S}$ as a real linear space (closed under the operation of taking the absolute value) of real continuous functions on a compact Hausdorff space introduces some difficulty here. We abbreviate those portions of the proof which are duplications of the argument in Theorem 1.

LEMMA.

$$W_2 \geqq W_1 \geqq 0,\ 0 < \alpha_2 \leq \alpha_1 \Longrightarrow (W_2)(\alpha_2 + W_2)^{-1} \geqq (W_1)(\alpha_1 + W_1)^{-1}.$$

*Proof.* From our hypothesis and the fact that all the expressions are only functions of $W_2$ we apply [1, page 201] and deduce that

$$(W_2)(\alpha_2 I + W_2)^{-1} \geqq (W_2)(\alpha_1 I + W_2)^{-1}.$$

We notice that $(z)(\alpha_1 + z)^{-1}$ is a "positive" [4, page 209] function, i. e. in the upper half of the complex plane the imaginary part of $(z)(\alpha_1 + z)^{-1}$ is non-negative. By [4, page 215], $(x)(\alpha_1 + x)^{-1}$ is a montone matrix function of arbitrarily high order in the interval $0 \leqq x < \infty$. By continuity considerations we deduce that $x(\alpha_1 + x)^{-1}$ is a monotone function for bounded positive semi-definite operators in Hilbert space and so $(W_2)(\alpha_1 + W_2)^{-1} \geqq (W_1)(\alpha_1 + W_1)^{-1}$. Our conclusion $(W_2)(\alpha_2 + W_2)^{-1} \geqq (W_1)(\alpha_1 + W_1)^{-1}$ now follows.[1]

THEOREM 2. *If $\mathfrak{S}_0$, the set of hermitian elements of $\mathfrak{A}_0$, which is a $C^*$ algebra without a unit, is lattice ordered, then $\mathfrak{A}_0$ is commutative.*

*Proof.* We consider the positive functionals $\omega \mid \mathfrak{S}_0$, where now $\| \omega \| = \sup\{\omega(UU^*) \mid \| U \| \leqq 1\}$. As before $\Omega_0$ is the set of positive functionals (states) having norm 1 and $\Pi_0$ is the set of extreme points (pure states) of $\Omega_0$. We wish now to show that $\mathfrak{A}_0$ is isometric, linearly isomorphic and order isomorphic to a space $\mathfrak{C}(Q_0)$, i. e. a real linear space of real

[1] The referee has suggested the following abbreviated proof which is similar to one communicated to the author by Professor J. W. Givens: "This fact can readily be proved directly (without the use of Loewner's result or continuity considerations). It is readily seen to suffice to show that if $A$ and $B$ are bounded self-adjoint operators on a Hilbert space with bounded inverses and such that $A \geqq B > 0$, then $B^{-1} \geqq A^{-1}$. Now for any bounded operator $C$, $A \geqq B$ implies that $C^*AC \geqq C^*BC$, and taking $C^{-1}$ to be a self-adjoint square root of $B$, we get $CAC \geqq I$, whence $C^{-1}A^{-1}C^{-1} \leqq I$, so $A^{-1} \leqq C^2 = B^{-1}$."

continuous functions on $Q_0$, the $w^*$ closure of $\Pi_0$, which has the property that if $\phi$ is an element of $\mathfrak{C}(Q_0)$, then $|\phi|$, the function which for each $\omega \varepsilon Q_0$ takes the value $|\phi(\omega)|$, is an element of $\mathfrak{C}(Q_0)$.

We readily verify that $\mathcal{A}_0$ satisfies Axioms I-IV of [3]. We now consider Axiom VI: $\|\,|U|\,\| = \|U\|$. Since $-|U| \leqq U \leqq |U|$ it follows that $\|U\| \leqq \|\,|U|\,\|$. From [1, page 201] we see that we can imbed $U$ in $\mathcal{A}(U)$, the closed subring generated by $U$ and $I$ in $\mathcal{A}$, the $C^*$ algebra obtained by adding a unit to $\mathcal{A}_0$. Since $\mathcal{A}(U)$ is a lattice we observe that $U^+ = \sup\{U, 0\}$ in $\mathcal{A}(U)$ and $U^- = \sup\{-U, 0\}$ in $\mathcal{A}(U)$ exist. Moreover $U^+ \varepsilon \mathcal{A}_0$, $U^- \varepsilon \mathcal{A}_0$, $U^+ \geqq U_+$, $U^- \geqq U_-$. Thus $U^+ + U^- \geqq U_+ + U_- = |U| \geqq 0$. But $\|U\| = \|U^+ + U^-\| \geqq \|\,|U|\,\| \geqq 0$, and so $\|U\| = \|\,|U|\,\|$, i.e., Axiom VI is satisfied.

In order to verify Axiom V:

$$U \geqq 0, \quad V \geqq 0 \Longrightarrow \|\sup\{U, V\}\| = \sup\{\|U\|, \|V\|\},$$

we define with [6, page 76], $\mu = \{U_1, \cdots, U_m\}$, $W(\mu) = \sum_{i=1}^{m} U_i U^*_i$, and $V(\mu) = W(\mu)(m^{-1}I + W(\mu))^{-1}$. The $\mu$'s form a system directed by set inclusion. From the Lemma, $\mu_\alpha \subset \mu_\beta \Longrightarrow V(\mu_\alpha) \leqq V(\mu_\beta) \leqq I$, where our elements are considered as elements of $\mathcal{A}$. Let

$$t(U;\mu) = \inf\{t \geqq 0 \mid -tV(\mu) \leqq U \leqq tV(\mu)\},$$

where if no real $t$ satisfying the inequality exists, we let $t(U;\mu) = +\infty$. If $\mu_\alpha \subset \mu_\beta$, then $t(U;\mu_\alpha) \geqq t(U;\mu_\beta) \geqq \|U\|$. If $U \geqq 0$, let $\mu = \{nU\}$, where $n$ is a positive integer. Then

$$0 \leqq U \leqq (n^{-1} + \|U\|)(nU)(1 + nU)^{-1}.$$

Thus $\|U\| \leqq t(U;\mu) \leqq n^{-1} + \|U\|$ and $\lim_\mu t(U;\mu) = \|U\|$ for $U \geqq 0$. Now if $U \geqq 0$ and $V \geqq 0$ and $\epsilon > 0$ there exist $\mu_1(\epsilon)$ and $\mu_2(\epsilon)$ such that

$$0 \leqq U \leqq (\|U\| + \epsilon)V(\mu_1(\epsilon)), \qquad 0 \leqq V \leqq (\|V\| + \epsilon)V(\mu_2(\epsilon)).$$

If we let $\mu_3(\epsilon) = \mu_1(\epsilon)U\mu_2(\epsilon)$, then

$$0 \leqq \sup\{U, V\} \leqq (\sup\{\|U\|, \|V\|\} + \epsilon)V(\mu_3(\epsilon)).$$

From this $\|\sup\{U, V\}\| \leqq \sup\{\|U\|, \|V\|\}$. Since the reverse inequality is trivial we see that Axiom V is verified.

Now that Axioms I-VI of [3] are satisfied by $\mathcal{A}_0$, we deduce from [3, Theorem 3] and its proof that with each $U \varepsilon \mathcal{A}_0$ we may associate a $\phi_U \varepsilon C(Q_0, R)$ where $Q_0$, the $w^*$ closure of $\Pi_0$, is compact and the mapping $U \to \phi_U$ is a one-one linear mapping into. Moreover

1′) $U \leq V \Longleftrightarrow \phi_U(\omega) \leqq \phi_V(\omega)$ for each $\omega \varepsilon Q_0 \Longleftrightarrow \phi_U(\omega) \leqq \phi_V(\omega)$ for each $\omega \varepsilon \Pi_0$,

2′) $\| U \| = \sup\{| \phi_U(\omega)| \mid \omega \varepsilon Q_0\} = \sup\{| \phi_U(\omega)| \mid \omega \varepsilon \Pi_0\}$,

3′) $\sup\{\phi_{UU^*}(\omega) \mid \| U \| \leq 1\} = 1$,

4′) $\phi \varepsilon \{\phi_U \mid U \varepsilon \mathfrak{H}_0\} \Longrightarrow | \phi | \varepsilon \{\phi_U \mid U \varepsilon \mathfrak{H}_0\}$,

where $| \phi |(\omega) = | \phi(\omega)|$ for each $\omega \varepsilon \Pi_0$ $(\omega \varepsilon Q_0)$.

Property 4′) is important since it implies that

$$\phi \varepsilon \{\phi_U \mid U \varepsilon \mathfrak{H}_0\} \Longrightarrow (\phi + | \phi |)2^{-1} \varepsilon \{\phi_U \mid U \varepsilon \mathfrak{H}_0\}.$$

Note that $\phi_{U_+}(\omega) = ((\phi + | \phi |)2^{-1})(\omega)$.

For each $\omega \varepsilon \Pi_0$, let $\mathfrak{J}_0 = \{U \mid U \varepsilon \mathcal{A}_0, \overline{\omega}_0(U) = 0\}$. Just as before $\mathfrak{J}_0$ is a closed two sided ideal (now in $\mathcal{A}_0$). A crucial fact in the proof is that $U \varepsilon \mathfrak{J}_0 \Longrightarrow U = U_+ - U_-$ where $U_+ \geqq 0$, $U_- \geqq 0$, $U_+ \varepsilon \mathfrak{J}_0$, and $U_- \varepsilon \mathfrak{J}_0$. We take care of this by using the characterization of $\phi_{U_+}$ given in the last paragraph. $\mathfrak{J}_0$ is also a two-sided ideal in $\mathcal{A}$ ($\mathcal{A}_0$ augmented in standard fashion with $I$). As before we deduce that $\overline{\omega}_0 : V \to \overline{\omega}_0(V)$ defines a homomorphism from $\mathcal{A}$ to the complex numbers and that for each $\omega_0 \varepsilon \Pi_0$, $\overline{\omega}_0(UV) = \overline{\omega}_0(U)\ \overline{\omega}_0(V) = \overline{\omega}_0(VU)$. Since $\Pi_0$ is dense in $Q_0$, we infer that $\phi_{UV} = \phi_{VU}$ and $UV = VU$. Thus Theorem 2 is proved.

If we examine the proofs of Theorems 1 and 2 we notice that we could dispense with completeness. The relevant representation theorems without completeness hypotheses can be found in [2, 3]. The intermediate constructions we make using completeness cause no essential difficulty.

COROLLARY. If $\mathcal{A}$ is a self-adjoint algebra (with or without unit, no completeness assumption) of bounded operators on a complex Hilbert space and if $\mathfrak{H}$, the set of hermitian elements of $\mathcal{A}$, is a lattice under our ordering, then $\mathcal{A}$ is commutative.

Let $\mathcal{A}^n$ be the $C^*$ algebra of all linear operators on $H^n$, n-dimensional complex euclidean space, and let $\mathfrak{H}^n$ be the set of hermitian elements of $\mathcal{A}^n$. We are concerned with the possibility that if $\mathfrak{H}$, a real linear subspace of $\mathfrak{H}^n$, were a lattice under its natural ordering, then $\mathfrak{H}$ would have to be a real commutative algebra. The facts are as follows. When $n = 2$ and $\mathfrak{H}$ does not contain a unit, then $\mathfrak{H}$ need not be a commutative algebra. For instance, choose a coordinate system and let $\mathfrak{H}$ be the real linear subspace formed by the matrices $\{A_\alpha \mid -\infty < \alpha < \infty\}$, where

$$A_\alpha = \begin{pmatrix} \alpha & 0 \\ 0 & 2\alpha \end{pmatrix}.$$

When $n = 2$ and $\mathfrak{S}$ contains the unit then $\mathfrak{S}$ is a real commutative algebra. When $n \geq 3$, $\mathfrak{S}$ need not be a real commutative algebra even though $\mathfrak{S}$ contains the identity operator. For example let $n = 3$ and after choosing a coordinate system let $\mathfrak{S}$ be the real linear space represented by matrices of the form $\{\alpha A + \beta B \mid -\infty < \alpha, \beta < \infty\}$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathfrak{S}$ is not an algebra since $AB \notin \mathfrak{S}$.

It is readily checked that $\alpha A + \beta B$ is positive semi-definite if and only if $\alpha \geqq 0$, $\beta \geqq 0$. This makes $\mathfrak{S}$ a well-known vector lattice.

*Remark.* Conversations with Professor Lipman Bers on the proof of the known fact that the collection of all real symmetric $2 \times 2$ matrices is not a lattice, led to the technique for constructing the counter-examples above and also raised the questions which they answer.

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